MATH 4030 Differential Geometry
Homework 7
due 28/10/2015 (Wed) at 5PM

Problems

You can directly quote results from previous Homeworks. Einstein summation convention is used throughout the assignment.

1. Let $f : U \to \mathbb{R}^3$ be a regular parametrized surface. Prove the Gauss and Weingarten formula:
$$\frac{\partial^2 f}{\partial u^i \partial u^j} = \Gamma_{ij}^k \frac{\partial f}{\partial u^k} - h_{ij}\nu,$$
$$\frac{\partial \nu}{\partial u^i} = g^{jk} h_{ij} \frac{\partial f}{\partial u^k}.$$

2. Let $f(t, \varphi) = (r(t)\cos\varphi, r(t)\sin\varphi, h(t))$, $t \in (a, b)$, $\varphi \in \mathbb{R}$, be a regular parametrized surface of revolution around the z-axis such that $r(t) > 0$ and $r'(t)^2 + h'(t)^2 > 0$ for all t. Show that for any fixed $\varphi_0 \in \mathbb{R}$, the curve given by $t \mapsto f(t, \varphi_0)$ is a geodesic on the surface after reparametrization of the curve. What about the curves $\varphi \mapsto f(t_0, \varphi)$ for some t_0 fixed? Prove your claim or give a counterexample.

3. Consider the upper half plane $\mathbb{R}^2_+ := \{(x, y) \in \mathbb{R}^2 : y > 0\}$ endowed with the following metric, i.e. first fundamental form,
$$(g_{ij}) = \frac{1}{y^2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$
   (a) Compute the Christoffel symbols Γ_{ij}^k in the standard coordinates x, y.
   (b) Describe *all* the geodesics for this intrinsic surface $(\mathbb{R}^2_+, g)$.
   (c) Show that for any given complete geodesic ℓ and a point $p \notin \ell$, there exists infinitely many complete geodesics through p which is disjoint from ℓ (*Note: A geodesic is complete if it can be arc-length parametrized by some* $c(s)$ *where* $s \in (-\infty, \infty)$).

Suggested Exercises

1. (Kühnel Ch.4 Q.3) Suppose c is a curve lying on an embedded submanifold $\Sigma \subset \mathbb{R}^3$. Let $p \in \Sigma$ and that $c(0) = p$. Show that the geodesic curvature κ_g of c at p conincides with the curvature κ of the plane curve which is obtained as the orthogonal projection of c in the tangent plane $T_p\Sigma$.

2. (Kühnel Ch.4 Q.4) Show that (locally) a curve c on a regular parametrized surface is uniquely determined by the geodesic curvature as a function of the arc length s, if one prescribes a point $c(0)$ and the direction $c'(0)$. Compare this with the planar case and the case that c is a geodesic.

3. (Kühnel Ch.4 Q.5) Show that a Frenet curve on a regular parametrized surface is a geodesic if and only if the unit normal to the surface coincides with the principal normal of the curve up to a sign.