MATH 4030 Differential Geometry
Homework 6
due 20/10/2015 (Tue) at 5PM

Problems

You can directly quote results from previous Homeworks.

1. Show that the area of a regular parametrized surface $f(u_1, u_2) : U \to \mathbb{R}^3$ can be calculated by the formula
$$\int_U \left\| \frac{\partial}{\partial u_1} \times \frac{\partial}{\partial u_2} \right\| du_1 du_2.$$
Hint: Fix some orthonormal basis $\{v_1, v_2\}$ *for the tangent plane.*

2. Let $F : \mathbb{R}^3 \to \mathbb{R}$ be a smooth function. A point $p \in \mathbb{R}^3$ is said to be a *critical point of* F if $DF(p) = 0$. A real number $c \in \mathbb{R}$ is said to be a *critical value of* F if there exists some $p \in F^{-1}(c)$ which is a critical point of F.
   (a) Show that $\Sigma := F^{-1}(c) \subset \mathbb{R}^3$ is an embedded submanifold if c is not a critical value of F.
   (b) Prove that the embedded submanifold Σ in (a) is orientable.

3. Consider a surface of revolution as in Q.2 of Homework 5:
   (a) Can you give explicit examples of the curve $(x(t), z(t))$ such that the surface of revolution has constant Gauss curvature K for (i) $K > 0$, (ii) $K = 0$ and (iii) $K < 0$? Justify your answer.
   (b) Can you give explicit examples of the curve $(x(t), z(t))$ such that the surface of revolution has constant mean curvature H for (i) $H > 0$, (ii) $H = 0$ and (iii) $H < 0$? Justify your answer.

4. Let $f : U \to \mathbb{R}^3$ be a regular parametrized surface. For each positive real number $\lambda > 0$, define another regular parametrized surface $\tilde{f} : U \to \mathbb{R}^3$ by $\tilde{f}(u_1, u_2) = \lambda f(u_1, u_2)$.
   (a) Prove that the Gauss and mean curvatures of f and $\tilde{f}$ satisfies the following scaling properties:
$$\tilde{K} = \frac{1}{\lambda^2} K \qquad \text{and} \qquad \tilde{H} = \frac{1}{\lambda} H.$$
   (b) Show that the quantity $\int (H^2 - K)\, dA$ is invariant under rescaling, i.e.
$$\int_U (H^2 - K) \sqrt{\det(g_{ij})}\, du_1 du_2 = \int_U (\tilde{H}^2 - \tilde{K}) \sqrt{\det(\tilde{g}_{ij})}\, du_1 du_2.$$
   (c) Can you give two different (i.e. not related by a rigid motion in $\mathbb{R}^3$) examples of a a regular parametrized surface with constant mean curvature $H = 4$? Justify why they are *different*.

5. Determine all the umbilic points on the ellipsoid (where $a > b > c > 0$ are distinct positive real numbers):
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$
At which point(s) does the Gauss curvature K attains its maximum? What about for the mean curvature H (assuming the orientation is chosen such that $H > 0$)?

6. Find the geodesic curvature κ_g and normal curvature κ_ν for the circle $C_\alpha := \mathbb{S}^2 \cap \{z = \alpha\}$ as a regular curve lying on the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$. *Hint: Your answer should depend on* $\alpha \in (-1, 1)$.

Suggested Exercises

1. (Kühnel Ch.3 Q.4) Show that at a fixed point p on a regular parametrized surface, the mean curvature is equal to (twice of) the integral mean of all normal curvatures, i.e.
$$H = \frac{1}{\pi} \int_0^{2\pi} \kappa_\nu(\varphi)\, d\varphi.$$
Here we view κ_ν as a function of the angle φ, which parametrizes the set of unit tangent vectors at this point.

2. (Kühnel Ch.3 Q.15) Show that the catenoids are the only non-flat minimal surface which is a surface of revolution.

3. (do Carmo P. 151) Let $f : U \to \mathbb{R}^3$ be a regular parametrized surface with Gauss map $\nu : U \to \mathbb{S}^2$. Suppose $c(s) : (a, b) \to U$ is a regular parametrized curve.
   (a) Show that if the curve c does not pass through any parabolic points of f, then the curve $\nu \circ c$ is a regular parametrized curve lying on the sphere $\mathbb{S}^2$.
   (b) Suppose c is a line of curvature for the parametrized surface f. If $\kappa(t)$ is the curvature of $f \circ c$ at t as a space curve in $\mathbb{R}^3$, then
$$\kappa = |\kappa_n \kappa_S|,$$
   where κ_n is the normal curvature of the parametrized surface f along the direction of $(f \circ c)'(t)$ at the point $(f \circ c)(t)$ and κ_S is the curvature of $\nu \circ c$ at t as a space curve in $\mathbb{R}^3$.