MATH 4030 Differential Geometry
Homework 5
due 15/10/2015 (Thur) at 5PM

Problems

You can use some of your results from Homework 4.

1. Let $u : U \to \mathbb{R}$ be a smooth function defined on an open set $U \subset \mathbb{R}^2$. Consider the regular parametrized surface $f : U \to \mathbb{R}^3$ defined by

$$f(x, y) := (x, y, u(x, y)).$$

   (a) Compute the second fundamental form h as a symmetric 2×2 matrix.

   (b) Find the Gauss curvature K.

   (c) Find the mean curvature H. Show that the surface is minimal (i.e. $H \equiv 0$) if and only if the function u satisfies the minimal surface equation:

$$(1 + u_x^2)u_{yy} - 2u_x u_y u_{xy} + (1 + u_y^2)u_{xx} = 0.$$

   (d) Show that $u : (-\pi/2, \pi/2) \times (-\pi/2, \pi/2) \to \mathbb{R}$ defined by

$$u(x, y) = \log \frac{\cos x}{\cos y}$$

   satisfies the minimal surface equation. (The graph is called the *Scherk minimal surface*.)

2. Let $(x(t), z(t)) : (a, b) \to \mathbb{R}^2$ be an arc-length parametrized curve in the xz-plane such that $x(t) > 0$ for all $t \in (a, b)$. Consider the rotationally invariant surface parametrized by $f : (a, b) \times \mathbb{R} \to \mathbb{R}^3$

$$f(t, \varphi) = (x(t) \cos \varphi, x(t) \sin \varphi, z(t)).$$

   (a) Compute the second fundamental form h as a symmetric 2×2 matrix.

   (b) Show that the Gauss curvature is given by $K = -\frac{x''}{x}$.

   (c) Show that the mean curvature is given by $H = -\frac{xz''+x'z'}{xx'}$. Show that the surface is minimal (i.e. $H \equiv 0$) if and only $xz' \equiv$ constant.

   (d) Show that the *catenoid* generated by the curve $x = \cosh z$, $z \in \mathbb{R}$ is a minimal surface.

3. Let $a > b > 0$ be two positive constants. Consider the torus parametrized by $f : (0, 2\pi)\times(0, 2\pi) \to \mathbb{R}^3$

$$f(u, v) = ((a + b \cos u) \cos v, (a + b \cos u) \sin v, b \sin u).$$

   (a) Find the total area of the torus.

(b) Find the Gauss curvature K and mean curvature H.

(c) Find the total squared mean curvature (also called the *Willmore energy*) $\int H^2\, dA$ of the torus *Hint: The answer depends on a and b.*

(d) Can you find a pair of constants a and b such that the Willmore energy in (c) is minimized?

4. Let $f : U \to \mathbb{R}^3$ be a regular parametrized surface and suppose $\varphi : \tilde{U} \to U$ is an orientation preserving (i.e. $\det(D\varphi) > 0$) diffeomorphism between two open sets $\tilde{U}, U \subset \mathbb{R}^2$. Let $\tilde{f} := f \circ \varphi : \tilde{U} \to \mathbb{R}^3$.

   (a) Let $\nu : U \to \mathbb{S}^2$ and $\tilde{\nu} : \tilde{U} \to \mathbb{S}^2$ be the Gauss maps defined by the parametrizations f and $\tilde{f}$ respectively. Show that $\tilde{\nu} = \nu \circ \varphi$.

   (b) Show that the shape operator is invariant in the sense that the following diagram of linear maps commutes:

$$\begin{tikzcd} T_u f \arrow[r, "S"] \arrow[d, "\cong"'] & T_u f \arrow[d, "\cong"] \\ T_{\tilde{u}} \tilde{f} \arrow[r, "\tilde{S}"] & T_{\tilde{u}} \tilde{f} \end{tikzcd}$$

   where $\varphi(\tilde{u}) = u$ and the vertical isomorphisms are given by $(D_{\tilde{u}} \tilde{f}) \circ (D_u \varphi^{-1}) \circ (D_u f)^{-1}$ and its inverse. This implies that the Gauss and mean curvatures are invariantly defined, i.e. $\tilde{K} = K \circ \varphi$ and $\tilde{H} = H \circ \varphi$.

Suggested Exercises

1. (Kühnel Ch.3 Q.17) Let $f : U \to \mathbb{R}^3$ be a regular parametrized surface with Gauss map $\nu : U \to \mathbb{S}^2$. We define for each $\epsilon \in \mathbb{R}$ small the family of *parallel surfaces at distance* ϵ by

$$f_\epsilon(u_1, u_2) := f(u_1, u_2) + \epsilon\, \nu(u_1, u_2).$$

   Assume that f_ϵ is still a regular parametrized surface. Show that

   (a) The principal curvatures of f_ϵ and f have a ratio of

$$\kappa_i^{(\epsilon)} = \frac{\kappa_i}{1 + \epsilon \kappa_i}.$$

   (b) In case f has constant mean curvature $H \neq 0$, f_ϵ has constant Gauss curvature for $\epsilon = -1/H$.

2. (do Carmo P.151 Q.8) Describe the region of $\mathbb{S}^2$ covered by the image of the Gauss map of the following surfaces:

   (a) Paraboloid of revolution $z = x^2 + y^2$.

   (b) Hyperboloid of revolution $x^2 + y^2 - z^2 = 1$.

   (c) Catenoid $x^2 + y^2 = \cosh^2 z$.