**MATH 4030 Differential Geometry**
**Homework 4**
*due 6/10/2015 (Tue) at 5PM*

# Problems

1. Let $h : U \to \mathbb{R}$ be a smooth function defined on an open set $U \subset \mathbb{R}^2$. Consider the parametrized surface $f : U \to \mathbb{R}^3$ defined by
$$f(x, y) := (x, y, h(x, y)).$$
   (a) Show that $f$ is regular everywhere on $U$.
   (b) Find a unit normal vector field along $f$.
   (c) Compute $2 \times 2$ matrix $(g_{ij})$ of the first fundamental form of the parametrized surface $f$.

2. A parametrization $f(u_1, u_2) : U \to \mathbb{R}^3$ is *conformal* if there exists a smooth positive function $\lambda : U \to \mathbb{R}$ such that the first fundamental form associated to $f$ at any $u \in U$ has the form
$$(g_{ij}) = \lambda^2(u) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$
   (a) Show that the differential $Df_u$ of a conformal parametrization $f$ is a linear map which preserves the angles between any two vectors.
   (b) Is the spherical coordinates $f(\varphi, \theta) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$ a conformal parametrization of the unit sphere $\mathbb{S}^2$ (without the north and south poles)?
   (c) Prove that the *Mercator projection* defined by
$$f(u, \varphi) = \frac{1}{\cosh u}(\cos\varphi, \sin\varphi, \sinh u),$$
   is also a conformal parametrization of the unit sphere without the north and south poles.

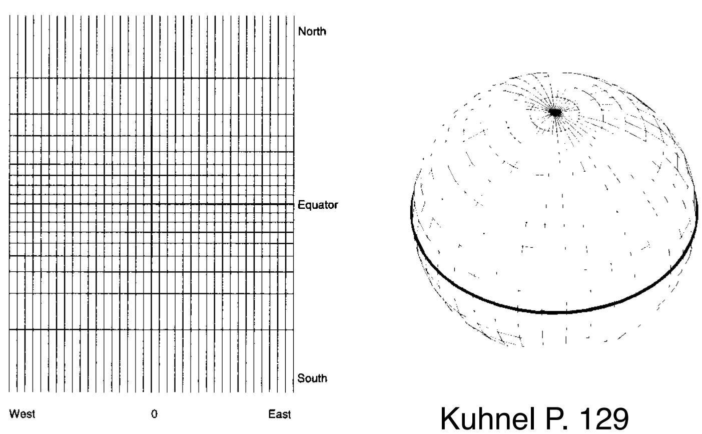


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**Figure 3.30.** Coordinate grid of the Mercator projection

3. Consider the torus of revolution given by the parametrization $f : (0, 2\pi) \times (0, 2\pi) \to \mathbb{R}^3$ defined by

$$f(u, v) = ((a + b\cos u)\cos v, (a + b\cos u)\sin v, b\sin u).$$

   (a) Show that $f$ is regular everywhere.

   (b) Find an outward pointing unit normal vector field along $f$.

   (c) Compute the first fundamental form associated to $f$ as a $2 \times 2$ matrix $(g_{ij})$.

   (d) Is $f$ a conformal parametrization?

4. Let $c(t) = (x(t), z(t)) : (a, b) \to \mathbb{R}^2$ be a regular parametrized curve in the $xz$-plane such that $x(t) > 0$ for all $t \in (a, b)$. Define a parametrized surface $f : (a, b) \times \mathbb{R} \to \mathbb{R}^3$ by

$$f(t, \varphi) = (x(t)\cos\varphi, x(t)\sin\varphi, z(t)).$$

   (a) Show that $f$ is regular everywhere.

   (b) Find a unit normal vector field on the parametrized surface $f$.

   (c) Compute the first fundamental form associated to $f$ as a $2 \times 2$ matrix $(g_{ij})$.

   (d) Show that $f$ is conformal if and only if $x'(t)^2 + z'(t)^2 = x(t)^2$ for all $t \in (a, b)$.

   (e) Prove that there exists a reparametrization $\tilde{f}$ of $f$ which is conformal.

## Suggested Exercises

1. (Kühnel Ch.3 Q.6) Let $f : U = [0, A] \times [0, B] \to \mathbb{R}^3$ be a regular parametrized surface. Show that the following conditions (i) and (ii) are equivalent:

   (i) For each rectangle $R = [u_1, u_1 + a] \times [u_2, u_2 + b] \subset U$, the opposite sides of $f(R)$ are of equal length.

   (ii) One has $\frac{\partial g_{11}}{\partial u_2} = \frac{\partial g_{22}}{\partial u_1} = 0$ in all of $U$.

   The coordinate grid (or two-parameter family of curves) formed by the $u_1$ and the $u_2$ lines is called a *Tchebychev grid*. Show that under these conditions there is a reparametrization $\tilde{f} = f \circ \varphi$, $\varphi : \tilde{U} \to U$, such that the first fundamental form can be written as

$$(\tilde{g}_{ij}) = \begin{pmatrix} 1 & \cos\vartheta \\ \cos\vartheta & 1 \end{pmatrix},$$

   where $\vartheta$ is the angle between the coordinate lines. *Hint: Set* $\varphi^{-1}(u_1, u_2) = (\int \sqrt{g_{11}}du_1, \int \sqrt{g_{22}}du_2.)$

2. *Inverse Function Theorem*: Let $f : U \subset \mathbb{R}^n \to \mathbb{R}^n$ be a $C^1$ map defined on an open subset $U \subset \mathbb{R}^n$. Suppose $a \in U$ the differential $Df_a$ is non-singular. Then, there exists an open set $V$ containing $a$ and an open set $W$ containing $f(a)$ such that the restriction $f : V \to W$ is a diffeomorphism (i.e. there exists some differentiable map $f^{-1} : W \to V$ such that $f \circ f^{-1} = id_W$ and $f^{-1} \circ f = id_V$). Prove the above statement following the steps below:

(i) Show that if the theorem is true for $Df_a = id$, then the theorem is true in general.

(ii) Show that there exists a closed rectangle $R$ containing $a$ in its interior such that

  (1) $f(x) \neq f(a)$ for all $x \in R$, $x \neq a$,

  (2) $Df_x$ is non-singular for all $x \in R$,

  (3) $|D_i f^j_x - D_i f^j_a| < 1/2n^2$ for all $x \in R$, $i, j = 1, \cdots, n$, where $D_i f^j$ is the $i$-th partial derivative of the $j$-th component of $f$.

(iii) Use the mean value theorem to prove that if $f : R \to \mathbb{R}^n$ is a $C^1$ map defined on a closed rectangle $A \subset \mathbb{R}^n$ such that $|D_i f^j_x| \leq M$ for all $x \in$int$(A)$, then

$$|f(x) - f(x')| \leq n^2 M |x - x'|, \qquad \text{for all } x, x' \in R.$$

(iv) Use (iii) and (iv) to show that $|x - x'| \leq 2|f(x) - f(x')|$ for all $x, x' \in R$.

(v) Let $0 < d := \min\{|f(x) - f(a)| : x \in \partial R\}$ and define $W := \{y : |y - f(a)| < d/2\}$. Show that for each $y \in W$, there is a unique $x \in R$ such that $f(x) = y$. *(Hint: Consider the minimum of the function $g(x) = |y - f(x)|^2$ on $R$.)* Hence, there exists an inverse $f^{-1} : W \to V$ where $V :=$int$(R) \cap f^{-1}(W)$.

(vi) Show that $f^{-1}$ is differentiable from definition.

3. Use the Inverse Function Theorem to prove the following *Implicit Function Theorem*: Suppose $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ is $C^1$ in an open set containing the point $(a, b)$ with $f(a, b) = 0$. Suppose that the $m \times m$ matrix $(D_{n+j} f^i(a, b))$, $1 \leq i, j \leq m$ is non-singular. Then, there exists an open set $A \subset \mathbb{R}^n$ containing $a$ and an open set $B \subset \mathbb{R}^m$ containing $b$ such that for each $x \in A$, there is a unique $g(x) \in B$ such that $f(x, g(x)) = 0$. Moreover, the map $x \mapsto g(x)$ is differentiable.

4. Let $F : \mathbb{R}^3 \to \mathbb{R}$ be a smooth function. Suppose that $c \in \mathbb{R}$ such that $DF_x \neq 0$ for all $x \in \mathbb{R}^3$ with $F(x) = c$. Show that the level set $F^{-1}(c)$ can be locally expressed as a graph over some open sets of one of the coordinate planes. *Hint: Apply Implicit Function Theorem.*