MATH 4030 Differential Geometry
Homework 3
due 29/9/2015 (Tue) at 5PM

Problems

1. (do Carmo Sec. 1-7 Q.2) Let $\overline{AB}$ be the line segment joining two points A,B on the plane $\mathbb{R}^2$ and let $\ell >$ length of $\overline{AB}$. Show that the curve C joining A and B, with length ℓ, and such that together with $\overline{AB}$ bounds the largest possible area is an arc of a circle passing through A and B (Fig 1-35). (For simplicity, you may assume that C always lies on one side of the unique line passing through A and B.)

2. (do Carmo Sec. 1-7 Q.4) Let C be a plane curve and let T be the tangent line at a point $p \in C$. Draw a line L parallel to the normal line at p and at a distance d of p (Fig. 1-36). Let h be the length of the segment determined on L by C and T (thus, h is the "height" of C relative to T). Prove that
$$|\kappa(p)| = \lim_{d \to 0} \frac{2h}{d^2},$$
where $\kappa(p)$ is the curvature of C at p.

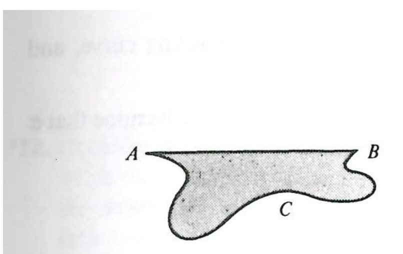

Figure 1-35

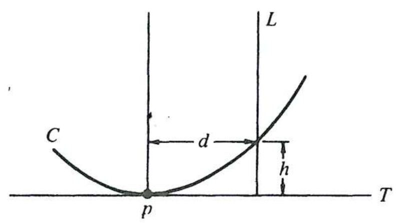

Figure 1-36

3. (do Carmo Sec. 1-7 Q.5) If a closed plane curve C is contained inside a disk of radius r, prove that there exists a point $p \in C$ such that the curvature $\kappa(p)$ of C at p satisfies $|\kappa(p)| \geq 1/r$.

4. (do Carmo Sec. 1-7 Q.6) Let $c(s), s \in [0, \ell]$ be a positively oriented closed convex regular plane curve parametrized by arc length. Let $\{e_1(s), e_2(s)\}$ be the Frenet frame of $c(s)$. The curve
$$\tilde{c}(s) := c(s) - re_2(s),$$
where $r > 0$ is a constant, is called a *parallel curve* to c. Show that

   (a) Length $(\tilde{c})$ = Length $(c) + 2\pi r$.

   (b) Area $(\tilde{\Omega})$ = Area $(\Omega) + r\ell + \pi r^2$, where Ω, $\tilde{\Omega}$ are the regions bounded by c and $\tilde{c}$ respectively.

   (c) $\tilde{\kappa}(s) = \kappa(s)/(1 + r\kappa(s))$ where κ, $\tilde{\kappa}$ are the curvatures of c and $\tilde{c}$ respectively.

5. (do Carmo Sec. 1-7 Q.13) Let C be an oriented plane closed curve with curvature $\kappa > 0$ everywhere. Assume that C has at least one point p of self intersection (Fig. 1-41). Prove that

   (a) There is a point $p' \in C$ such that the tangent line T' at p' is parallel to some tangent at p.

   (b) The rotation angle of the tangent in the positive arc of C made up by $pp'p$ is $> \pi$.

   (c) The rotation index of C is ≥ 2.

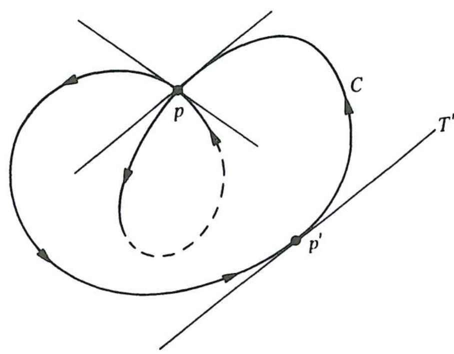

Figure 1-41