MATH 4030 Differential Geometry
Homework 2
due 23/9/2015 (Wed) at 5PM

Problems

1. (Kühnel Ch.2 Q.17) In the orthogonal (but not normal) three-frame $c', c'', c' \times c''$, prove that the Frenet equations of a space curve take the equivalent form

$$\begin{pmatrix} c' \\ c'' \\ c' \times c'' \end{pmatrix}' = \begin{pmatrix} 0 & 1 & 0 \\ -\kappa^2 & \frac{\kappa'}{\kappa} & \tau \\ 0 & -\tau & \frac{\kappa'}{\kappa} \end{pmatrix} \begin{pmatrix} c' \\ c'' \\ c' \times c'' \end{pmatrix}.$$

Hence, the entries of the matrix depend in some sense rationally (i.e. without roots) on $\kappa^2 = \langle c'', c'' \rangle$ and τ (because of the relation $\kappa'/\kappa = \frac{1}{2}(\log \kappa^2)'$).

2. (Kühnel Ch.2 Q.14) Show that the *osculating cubic parabola* of a Frenet curve c in $\mathbb{R}^3$, defined by

$$s \mapsto c(0) + se_1(0) + \frac{s^2}{2}\kappa(0)e_2(0) + \frac{s^3}{6}\kappa(0)\tau(0)e_3(0),$$

has at the point $s = 0$ the same curvature $\kappa(0)$ and torsion $\tau(0)$ as c itself.

3. (Kühnel Ch.2 Q.15) In spherical coordinates φ, ϑ, let a regular curve be given by the functions $(\varphi(s), \vartheta(s))$ inside the sphere with parametrization $(\cos\varphi\cos\vartheta, \sin\varphi\cos\vartheta, \sin\vartheta)$. For $s = 0$ the tangent to this curve is tangent to the equator $\vartheta = 0$, i.e. $\vartheta'(0) = 0$. Prove that the geodesic curvature is given by $\vartheta''(0) = \frac{d^2\vartheta}{ds^2}|_{s=0}$, and the curvature is consequently

$$\kappa(0) = \sqrt{1 + (\vartheta''(0))^2}.$$

4. (Kühnel Ch.2 Q.20) Show the following: (i) c is a helix if and only if the Darboux vector $\vec{D}$ is a constant vector; (ii) c is a slope line if and only if $\vec{D}/\|\vec{D}\|$ is constant.

5. (Kühnel Ch.2 Q.23) Let c be a Frenet curve in $\mathbb{R}^n$. Show that

$$\det(c', c'', \cdots, c^{(n)}) = \prod_{i=1}^{n-1}(\kappa_i)^{n-i}.$$

6. (do Carmo P.47 Q.3) Compute the curvature of the ellipse

$$x = a\cos t, \qquad y = b\sin t, \qquad 0 \le t \le 2\pi,\ a \neq b,$$

and show that it has exactly four vertices (i.e. points at which $\kappa' = 0$), namely, the points $(\pm a, 0)$ and $(0, \pm b)$.

Suggested Exercises

1. (Kühnel Ch.2 Q.16) Show that a slope line with $\tau \neq 0$ lies on a sphere if and only if an equation $\kappa^2(s) = (-A^2s^2 + Bs + C)^{-1}$ is satisfied for some constants A, B, C, where $A = \frac{\tau}{\kappa}$. Prove that a spherical slope line through a point on the equator can never reach the north pole. It ends at a point where it cuts a small circle around the north pole orthogonally.

2. (Kühnel Ch.2 Q.21) The axis of the accompanying screw-motion at a point $c(0)$ is the line in the direction of the Darboux vector $\vec{D}(0) = \tau(0)e_1(0) + \kappa(0)e_3(0)$ through the point

$$P(0) = c(0) + \frac{\kappa(0)}{\kappa^2(0) + \tau^2(0)} e_2(0).$$

Show that under these circumstances the tangent to the curve which passes through all of these points, namely,

$$P(s) = c(s) + \frac{\kappa}{\kappa^2 + \tau^2} e_2(s),$$

is proportional to $\vec{D}(s)$ if and only if $\kappa/(\kappa^2 + \tau^2)$ is constant.