MATH 4030 Differential Geometry
Homework 11
due 26/11/2015 (Thur) at 5PM

Problems

You can directly quote results from previous Homeworks.

1. (do Carmo P.282 Q.1) Let $\Sigma \subset \mathbb{R}^3$ be a closed orientable surface which is not homeomorphic to a sphere. Prove that there are points on Σ where the Gauss curvature K is positive, negative and zero respectively.

2. (do Carmo P.282 Q.2) Let T be a torus of revolution which can be parametrized (except along two curves) by

$$f(u, v) = ((2 + \cos u) \cos v, (2 + \cos u) \sin v, \sin u), \qquad 0 < u < 2\pi, 0 < v < 2\pi.$$

   Prove by an explicit calculation that

$$\iint_T K \, dA = 0.$$

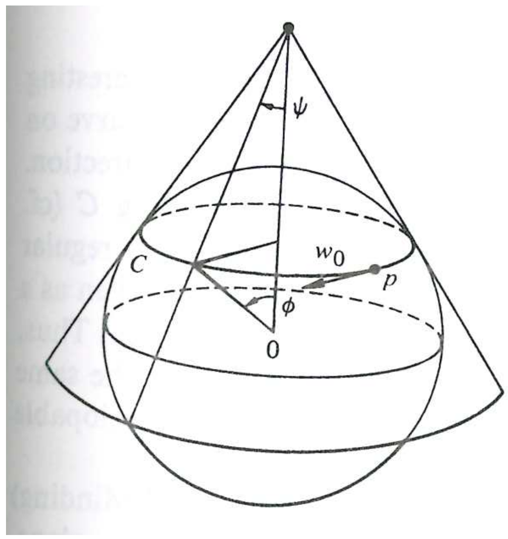
   Compute the Euler characteristic $\chi(T)$ for a torus by giving an explicit triangulation. Does the above result agree with the Gauss-Bonnet theorem?

3. (do Carmo P.282 Q.3) Let $\Sigma \subset \mathbb{R}^3$ be a closed orientable surface homeomorphic to a sphere and such that $K > 0$ everywhere. Let $\gamma \subset \Sigma$ be a simple closed geodesic in Σ and that $\Sigma \setminus \gamma = A \cup B$. Let $\nu : \Sigma \to \mathbb{S}^2$ be the Gauss map of Σ. Show that the images $\nu(A)$ and $\nu(B)$ in $\mathbb{S}^2$ have the same area.

4. (do Carmo P.282 Q.4) Compute the Euler characteristic of (a) an ellipsoid, (b) the surface $S := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^{10} + z^6 = 1\}$.

Suggested Exercises

1. (do Carmo P.282 Q.5) Let C be a parallel of colatitude φ on an oriented unit sphere $\mathbb{S}^2$, and let w_0 be a unit vector tangent to C at a point $p \notin C$. Take the parallel transport of w_0 along C and how that its position, after a complete turn, makes an angle $\Delta\varphi = 2\pi(1 - \cos\varphi)$ with the initial position w_0. Check that
$$\lim_{R \to p} \frac{\Delta\varphi}{A} = 1 = K_{\mathbb{S}^2},$$
where A is the area of the region R of $\mathbb{S}^2$ bounded by C.



2. (Kühnel Ch.5 Q.9) Let (U, g) be an abstract Riemannian surface and let $\Delta \subset U$ be a geodesic triangle which is the boundary of a simply connected domain. Show that the parallel translation along this boundary (traced through once) is a rotation in the tangent plane. Calculate the angle of rotation in terms of quantities which only depend on the interior of Δ.