MATH 4030 Differential Geometry
Homework 1
due 15/9/2015 (Tue) at 5PM

Problems

Prove all the following statements.

1. (Kühnel Ch.2 Q.4) A regular curve between two points p, q in $\mathbb{R}^n$ with minimal length is necessarily the line segment from p to q. *Hint: Consider the Schwarz inequality* $\langle X, Y\rangle \leq \|X\| \cdot \|Y\|$ *for the tangent vector and the difference vector* $q - p$.

2. (Kühnel Ch.2 Q.1) The curvature and the torsion of a Frenet curve $c(t)$ in $\mathbb{R}^3$ are given by the formulas
$$\kappa(t) = \frac{\|\dot{c} \times \ddot{c}\|}{\|\dot{c}\|^3} \quad \text{and} \quad \tau(t) = \frac{\mathrm{Det}(\dot{c}, \ddot{c}, \dddot{c})}{\|\dot{c} \times \ddot{c}\|^2}$$
for an arbitrary parametrization. For a plane curve we have
$$\kappa(t) = \frac{\mathrm{Det}(\dot{c}, \ddot{c})}{\|\dot{c}\|^3}.$$

3. (Kühnel Ch.2 Q.9-11) Let a plane curve be given in polar coordinates (r, φ) by $r = r(\varphi)$. Using the notation $r' = \frac{dr}{d\varphi}$, the arc length in the interval $[\varphi_1, \varphi_2]$ can be calculated as $s = \int_{\varphi_1}^{\varphi_2} \sqrt{r'^2 + r^2}\, d\varphi$, and the curvature is given by
$$\kappa(\varphi) = \frac{2r'^2 - rr'' + r^2}{(r'^2 + r^2)^{3/2}}.$$
Calculate the curvature of the curve given by $r(\varphi) = a\varphi$ (a is a constant), the so-called *Archimedean spiral*, see Figure 2.12. Moreover, show that (i) The length of the curve given in polar coordinates by $r(t) = \exp(t)$, $\varphi(t) = at$ with a constant a (the *logarithmic spiral*) in the interval $(-\infty, t]$ is proportional to the radius $r(t)$, see Figure 2.12. (ii) The position vector of the logarithmic spiral has a constant angle with the tangent vector.

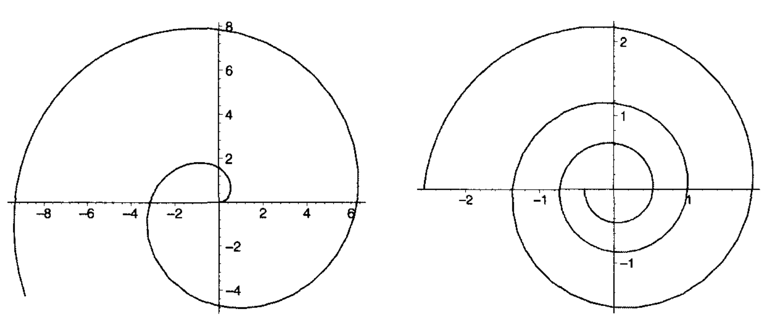

Figure 2.12. Archimedean spiral and logarithmic spiral

4. (Kühnel Ch.2 Q.6) If a circle of unit radius is rolled along a line (without friction), then a fixed point on that circle has its trajectory the so-called *cycloid*, see Figure 2.11. Find the arc-length parametrization $c(s) : [0, L] \to \mathbb{R}^2$ for one complete arch of the cycloid. Calculate its curvature $\kappa(s)$.

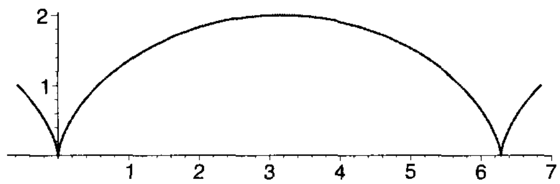

Figure 2.11. Cycloid

5. (Kühnel Ch.2 Q.3) Let $c(s)$ be a regular curve parametrized by arc length. If $\kappa(s) \neq 0$ for all s, then the *evolute* of c is defined to be the curve

$$\gamma(s) := c(s) + \frac{1}{\kappa(s)} e_2(s)$$

where $\{e_1(s), e_2(s)\}$ is the Frenet frame of $c(s)$ and $\kappa(s)$ is the (signed) curvature. Show that γ is regular precisely where $\kappa' \neq 0$, and that the tangent to γ at the point $s = s_0$ intersects the curve c at $s = s_0$ perpendicularly. Moreover, prove that the evolute of a cycloid (see Problem 4 above) is also a cycloid.

6. (Kühnel Ch.2 Q.7) Find the unique (up to rigid motions) arc-length parametrized plane curve $c(s) : (0, \infty) \to \mathbb{R}^2$ whose curvature is given by $\kappa(s) = s^{-1/2}$.

Suggested Exercises

1. (Kühnel Ch.2 Q.2) At every point p of a regular plane curve c with $c''(p) \neq 0$ (or, equivalently, $\kappa(p) \neq 0$) there is a parabola which has a point of third order contact with the curve at p. The point of contact is the vertex of the parabola if and only if $\kappa'(p) = 0$.

2. (Kühnel Ch.2 Q.5) If all tangent vectors to the curve $c(t) = (3t, 3t^2, 2t^3)$ are drawn from the origin, then their endpoints are on the surface of a circular cone with axis be the line $x - z = y = 0$.

3. (Kühnel Ch.2 Q.8) The Frenet two-frame of a plane curve with given curvature function $\kappa(s)$ can be described by the exponential series for the matrix

$$\begin{pmatrix} 0 & \int_0^s \kappa(t)dt \\ -\int_0^s \kappa(t)dt & 0 \end{pmatrix}.$$

It follows that

$$\begin{pmatrix} e_1(s) \\ e_2(s) \end{pmatrix} = \sum_{i=0}^{\infty} \frac{1}{i!} \begin{pmatrix} 0 & \int_0^s \kappa \\ -\int_0^s \kappa & 0 \end{pmatrix}^i.$$