MATH 4030 Differential Geometry
Lecture Notes Part 4
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Elementary tensor calculus

We will study in this section some basic multilinear algebra and operations on tensors.

Let V be an n-dimensional vector space over $\mathbb{R}$. From linear algebra we know that the space of all linear functionals on V forms a vector space itself, called the dual of V, i.e.

$$V^* = \mathrm{Hom}(V;\mathbb{R}) = \{f : V \to \mathbb{R} \text{ linear}\}.$$

Elements of V are called vectors while elements of V^* are called covectors. If $\{e_1, e_2, \cdots, e_n\}$ is a basis for V, then there is a corresponding dual basis $\{e_1^*, e_2^*, \cdots, e_n^*\}$ for V^* defined by the relationship

$$e_i^*(e_j) = \delta_{ij} = \begin{cases} 1 & \text{when } i = j, \\ 0 & \text{when } i \neq j. \end{cases}$$

Any $v \in V$ can be written uniquely as a linear combination $v = a_1 e_1 + \cdots + a_n e_n$, and it can be written as a *column vector*:

$$v = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}.$$

On the other hand, any covector $v^* \in V$ can be written uniquely as $v^* = \alpha_1 e_1^* + \cdots + \alpha_n e_n^*$. It can be expressed as a *row vector*:

$$v^* = \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n \end{pmatrix}.$$

The vector space V and its dual V^* have a natural (non-degenerate) pairing between them:

$$v^*(v) = \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = \alpha_1 a_1 + \alpha_2 a_2 + \cdots + \alpha_n a_n.$$

One important thing we learned from linear algebra is that although V and V^* are isomorphic vector spaces, they are not *canonically isomorphic*. In other words, there is no *default* isomorphism between V and V^*. Another example of this kind is that any n-dimensional real vector space V is isomorphic to $\mathbb{R}^n$ by taking a basis. However, the isomorphism $V \cong \mathbb{R}^n$ depends on the choice of the basis. To obtain a canonical isomorphism between V and V^*, one needs extra structure. For example, if $(V, \langle \cdot, \cdot \rangle)$ is an *inner product space*, then the linear map $\sharp : V \longrightarrow V^*$ defined by

$$v \mapsto v^\sharp := \langle v, \cdot \rangle$$

is an isomorphism which is independent of the choice of a basis (but depends on the inner product $\langle\cdot,\cdot\rangle$). Note that if $\{e_1, \cdots, e_n\}$ is an orthonormal basis for $(V, \langle\cdot,\cdot\rangle)$, then $\{e_1^\sharp, \cdots, e_n^\sharp\}$ is the dual basis for V^*. One also induces an inner product on V^* by requiring that $\sharp$ preserves their inner products.

Now, we define the tensor product of two real vector spaces V and W (not necessarily of the same dimension). The tensor product $V \otimes W$ is the set of all formal linear combinations of $v \otimes w$, $v \in V$ and $w \in W$, with an equivalence relation $\sim$ identifying elements through a *bilinear relation*:

$$V \otimes W = \text{span}\{v \otimes w : v \in V, w \in W\}/\sim,$$

where for any $c_1, c_2 \in \mathbb{R}$,

$$(c_1 v_1 + c_2 v_2) \otimes w = c_1(v_1 \otimes w) + c_2(v_2 \otimes w),$$

$$v \otimes (c_1 w_1 + c_2 w_2) = c_1(v \otimes w_1) + c_2(v \otimes w_2).$$

By convention, we set $V^{\otimes 0} = \mathbb{R}$. It is trivial that $V^{\otimes 1} = V$. To see what $V^{\otimes 2}$ is, if we take a basis $\{e_1, \cdots, e_n\}$, then we can write $v = \sum_{i=1}^n a_i e_i$, $w = \sum_{j=1}^n b_j e_j$, using the bilinear relations:

$$v \otimes w = \left(\sum_{i=1}^n a_i e_i\right) \otimes \left(\sum_{j=1}^n b_j e_j\right) = \sum_{i,j=1}^n a_i b_j (e_i \otimes e_j).$$

For example, $(e_1 + e_2) \otimes (e_1 - e_2) = e_1 \otimes e_1 + e_2 \otimes e_1 - e_1 \otimes e_2 - e_2 \otimes e_2$. Note that $e_1 \otimes e_2 \neq e_2 \otimes e_1$. In general, if $\{e_1, \cdots, e_n\}$ and $\{f_1, \cdots, f_m\}$ are bases for V and W respectively, then $\{e_i \otimes f_j\}$ forms a basis for $V \otimes W$. One can define higher tensor powers $V^{\otimes k}$ similarly. On the other hand, we have

$$V^* \otimes W \cong \text{Hom}(V, W).$$

One can identify an element $v^* \otimes w \in V^* \otimes W$ with a linear map $V \to W$ by

$$(v^* \otimes w)(v) := v^*(v) w,$$

and then extend to the rest of $V^* \otimes W$ by linearity. (*Exercise: Show that this is indeed an isomorphism.*)

Next, we define another operator called the *wedge product* $\wedge$. It shares similar multilinear properties but has the extra characteristic of being anti-symmetric. Let V be a real vector space as before. We define the k-th wedge product of V to be

$$\Lambda^k V = \text{span}\{v_1 \wedge v_2 \wedge \cdots \wedge v_k : v_i \in V\}/\sim,$$

where for any constant $c \in \mathbb{R}$,

$$v_1 \wedge \cdots \wedge (c v_i + w) \wedge \cdots \wedge v_k = c(v_1 \wedge \cdots \wedge v_i \wedge \cdots \wedge v_k) + (v_1 \wedge \cdots \wedge w \wedge \cdots \wedge v_k),$$

$$v_1 \wedge \cdots \wedge v_i \wedge v_{i+1} \wedge \cdots \wedge v_k = -v_1 \wedge \cdots \wedge v_{i+1} \wedge v_i \wedge \cdots \wedge v_k.$$

These two properties imply that the wedge product is linear in each variable and that $v_1 \wedge \cdots \wedge v_k = 0$ whenever $v_i = v_j$ for some $i \neq j$. If $\{e_1, \cdots, e_n\}$ is a basis for V, then it is easy to see that

$$\{e_{i_1} \wedge \cdots \wedge e_{i_k} : 1 \leq i_1 < i_2 < \cdots < i_k \leq n\}$$

forms a basis for $\Lambda^k V$. Hence, dim $\Lambda^k V = \binom{n}{k}$ where $n = dimV$.

By convention, $\Lambda^0 V = \mathbb{R}$. Also, $\Lambda^1 V = V$. For higher wedge products, for example, when $V = \mathbb{R}^2$ with standard basis $\{e_1, e_2\}$, then $\Lambda^2 V = \text{span}\{e_1 \wedge e_2\}$ and $\Lambda^k V = 0$ for all $k \geq 3$. When $V = \mathbb{R}^3$, we have

$$\begin{aligned} \Lambda^1 V &= \text{span}\{e_1, e_2, e_3\}, \\ \Lambda^2 V &= \text{span}\{e_1 \wedge e_2, e_1 \wedge e_3, e_2 \wedge e_3\}, \\ \Lambda^3 V &= \text{span}\{e_1 \wedge e_2 \wedge e_3\}. \end{aligned}$$

Note that $\Lambda^k V = 0$ whenever $k >$dim$V$ (*Exercise: Prove this.*)

The anti-symmetric property of wedge product is related to the determinant function when $k =$dim$V$.

Proposition 1. *Let $\{e_1, \cdots, e_n\}$ be a basis for V. For any $v_i = \sum_{i,j=1}^n a_{ij} e_j$ for $i = 1, \cdots, n$, we have*

$$v_1 \wedge \cdots \wedge v_n = \det(a_{ij})\, e_1 \wedge \cdots \wedge e_n.$$

Proof. Exercise to the reader. □

One can define the wedge product $\Lambda^k V^*$ for the dual space V^*, and in fact we have a canonical isomorphism $\Lambda^k V^* \cong (\Lambda^k V)^*$ defined by

$$(f_1 \wedge \cdots \wedge f_n)(v_1 \wedge \cdots \wedge v_n) = \det \begin{pmatrix} f_1(v_1) & \cdots & f_1(v_n) \\ \vdots & \ddots & \vdots \\ f_n(v_1) & \cdots & f_n(v_n) \end{pmatrix},$$

and extending to all the elements of $\Lambda^k V$ and $\Lambda^k V^*$ by linearity. For example,

$$(\begin{pmatrix} 2 & 3 \end{pmatrix} \wedge \begin{pmatrix} -1 & 2 \end{pmatrix}) \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \wedge \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right) = \det \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix} = 7.$$

An <u>orientation</u> of V is a choice of $[\eta]$ where $0 \neq \eta \in \Lambda^n V^*$ and $\eta \sim c\eta$ for $c > 0$. For example, the standard orientation on $\mathbb{R}^n$ is given by $[e_1^* \wedge e_2^* \wedge \cdots \wedge e_n^*]$. Note that since $\Lambda^n V^* \cong \mathbb{R}$, so there are only two distinct orientations on V. An ordered basis $\{v_1, \cdots, v_n\}$ for V is said to be <u>positive</u> if $\eta(v_1 \wedge \cdots \wedge v_n) > 0$. For example, if we take the standard orientation $\eta = e_1^* \wedge e_2^*$ of $\mathbb{R}^2$, then the ordered basis $\{e_1, e_2\}$ is positive while $\{e_2, e_1\}$ is negative.

Another useful operation is called <u>contraction</u>. Let $v \in V$ and $\omega \in \Lambda^k V^*$, then the contraction of ω with v, denoted by $\iota_v \omega \in \Lambda^{k-1} V^*$ is defined by

$$(\iota_v \omega)(v_1 \wedge \cdots \wedge v_{k-1}) = \omega(v \wedge v_1 \wedge \cdots \wedge v_{k-1}).$$

we have the following explicit formula for contraction:

$$\iota_v(v_1^* \wedge \cdots \wedge v_k^*) = \sum_{\ell=1}^{k} (-1)^{\ell-1} v_\ell^*(v)(v_1^* \wedge \cdots \wedge \widehat{v_\ell^*} \wedge \cdots \wedge v_k^*).$$

For example, if $v = ae_1 + be_2$, then $\iota_v(e_1^* \wedge e_2^*) = ae_2^* - be_1^*$. Hence, if we fix $v \in V$, then we have a sequence of contractions which reduces the *degree* of the wedge products:

$$0 \xleftarrow{\iota_v} \Lambda^0 V^* \xleftarrow{\iota_v} \Lambda^1 V^* \xleftarrow{\iota_v} \cdots \xleftarrow{\iota_v} \Lambda^n V^*.$$

Differential Forms in $\mathbb{R}^n$

In this section we introduce the language of differential forms in $\mathbb{R}^n$. Let U be an open domain in $\mathbb{R}^n$ equipped with the standard Euclidean metric $\langle \cdot, \cdot \rangle$. The tangent space at each $p \in U$ is denoted by T_pU, which is an n-dimensional real vector space. Note that T_pU is equipped with the inner product $\langle \cdot, \cdot \rangle$ so $(T_pU, \langle \cdot, \cdot \rangle)$ is in fact an inner product space. The tangent bundle is simply the disjoint union of all these tangent spaces:

$$TU := \coprod_{p \in U} T_pU \simeq U \times \mathbb{R}^n.$$

If we denote the standard coordinates on $\mathbb{R}^n$ by $x^1, \cdots, x^n$, then we obtain an orthonormal basis $\{ \frac{\partial}{\partial x^i}\big|_p, \cdots, \frac{\partial}{\partial x^n}\big|_p \}$ of T_pU. Because of this standard basis *globally* defined on U, we have a "canonical" identification of TU with $U \times \mathbb{R}^n$ once a coordinate system is fixed. A vector field X on U is just

$$X = \sum_{i=1}^{n} f_i \frac{\partial}{\partial x^i},$$

where $f_i : U \to \mathbb{R}$ are smooth functions,

Now, for each tangent space T_pU, we can take the dual vector space T_p^*U, which is called the cotangent space at p. As for the tangent bundle TU, we can put all the cotangent spaces together to form the cotangent bundle:

$$T^*U = \coprod_{p \in U} T_p^*U \simeq U \times \mathbb{R}^n.$$

We will denote the dual basis of $\{ \frac{\partial}{\partial x^i}\big|_p, \cdots, \frac{\partial}{\partial x^n}\big|_p \}$ in T_p^*U as $\{dx^1|_p, \cdots dx^n|_p\}$. We can take the ℓ-th wedge product of the cotangent spaces at each point as in the previous section and then put them all together to get the bundle $\Lambda^\ell T^*U$. A differential ℓ-form is just a section of $\Lambda^\ell T^*U$ which can be expressed as

$$\omega = \sum_{1 \le i_1 < \cdots < i_\ell \le n} f_{i_1 \cdots i_\ell} dx^{i_1} \wedge \cdots \wedge dx^{i_k},$$

for some smooth functions $f_{i_1 \cdots i_\ell} : U \to \mathbb{R}$.

For example, a 0-form is simply a smooth function $f : U \to \mathbb{R}$. When $n = 2$, the 1-forms have the form

$$\omega = f(x^1, x^2)dx^1 + g(x^1, x^2)dx^2,$$

and a 2-form would have the form

$$\omega = f(x^1, x^2)dx^1 \wedge dx^2.$$

We will use $\Omega^\ell(U)$ to denote the space of all ℓ-forms on U. Note that by the properties of wedge product, we have $\Omega^\ell(U) = 0$ whenever $\Omega \subset \mathbb{R}^n$ with $\ell > n$. The top form $dx^1 \wedge \cdots \wedge dx^n$ is called the volume form of $\mathbb{R}^n$, and we can integrate n-form on an n-dimensional domain $U \subset \mathbb{R}^n$ by

$$\int_U f(x)\, dx^1 \wedge \cdots \wedge dx^n := \int_U f(x)\, dx^1 \cdots dx^n,$$

whenever f is (Lebesgue) integrable on $U \subset \mathbb{R}^n$.

The power of the differential forms rests on the fact that we can do calculus on forms, i.e. we can differentiate and integrate forms. We have already seen how to integrate a top degree form above using the ordinary Lebesgue integral. We now discuss how to differentiate differential forms.

We will define an operation called the exterior derivative $d : \Omega^\ell(U) \to \Omega^{\ell+1}(U)$ for $\ell = 0, 1, 2, 3, \cdots$ which satisfies all the properties below:

(i) d is linear.

(ii) $df = \sum_{i=1}^n \frac{\partial f}{\partial x^i} dx^i$ for functions $f \in \Omega^0(U)$.

(iii) $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^p \omega \wedge d\eta$ where $\omega \in \Omega^p(U)$ and $\eta \in \Omega^\ell(U)$.

It is clear that the three properties above uniquely define the exterior derivative d. For example,

$$d\left(\sum_{I=(i_1,\cdots,i_\ell)} f_I(x)\, dx^{i_1} \wedge \cdots \wedge dx^{i_\ell} \right) = \sum_{I=(i_1,\cdots,i_\ell)} df_I \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_\ell}.$$

For example, when $n = 2$, we have

$$df = \frac{\partial f}{\partial x^1} dx^1 + \frac{\partial f}{\partial x^2} dx^2.$$

Therefore, df is equivalent to the vector field $\text{grad}(f)$ (under the dual pairing). On the other hand, the derivative of a 1-form is given by

$$d(f dx^1 + g dx^2) = df \wedge dx^1 + dg \wedge dx^2 = \left(\frac{\partial g}{\partial x^1} - \frac{\partial f}{\partial x^2} \right) dx^1 \wedge dx^2.$$

Note that the coefficient on the right hand side is just the rotation $\text{rot}(X)$ of the vector field $X = (f, g)$ in $\mathbb{R}^2$ which appears in Green's theorem. The reader is encourage to try out the case for $n = 3$ and see how it's related to the curl and div of vector fields in $\mathbb{R}^3$.

Theorem 2. $d^2 = d \circ d = 0$.

Proof. For $f \in \Omega^0(U)$, we see that

$$d^2 f = d\left(\sum_{i=1}^n \frac{\partial f}{\partial x^i} dx^i\right) = \sum_{i,j=1}^n \left(\frac{\partial^2 f}{\partial x^j \partial x^i}\right) dx^j \wedge dx^i = \sum_{1 \leq i < j \leq n} \left(\frac{\partial^2 f}{\partial x^i \partial x^j} - \frac{\partial^2 f}{\partial x^j \partial x^i}\right) dx^i \wedge dx^j = 0$$

since mixed partial derivatives are equal. That $d^2 = 0$ for general ℓ-forms follows similarly and is left as an exercise for the reader. □

The most important theorem on the calculus of forms is the fundamental theorem of calculus, which is called the generalized Stokes' Theorem.

Theorem 3 (Stokes' Theorem)**.** *Let $U \subset \mathbb{R}^n$ be a bounded smooth domain with boundary ∂U. Suppose $\omega \in \Omega^{n-1}(U)$, then we have*

$$\int_U d\omega = \int_{\partial U} \omega.$$

Moving frames on Surfaces

Now, we proceed to define differential forms on surfaces and use the idea of moving frames to prove the Gauss-Bonnet theorem in the next section. Since everything here is local, we can take a parametrized surface $f : U \to \mathbb{R}^3$ where $U \subset \mathbb{R}^2$ is a smooth bounded domain.

Suppose we have an orthonormal frame $\{X_1, X_2, X_3\}$ depending smoothly on $u \in U$ such that $X_1, X_2 \in T_u f$ and $X_3 \in N_u f$. Therefore, $X_3 = \nu$ is simply the unit normal to the surface while X_1 and X_2 are tangent to the surface. Note that in general we cannot take $X_1 = \frac{\partial f}{\partial u^1}$ and $X_2 = \frac{\partial f}{\partial u^2}$ as they cannot form an orthonormal basis *everywhere* unless the surface is flat. One way to obtain such a moving frame is that we can take

$$X_1 := \frac{\frac{\partial f}{\partial u^1}}{\|\frac{\partial f}{\partial u^1}\|}, \qquad X_2 := \nu \times X_1, \qquad X_3 := \nu.$$

It is easy to see that these give a moving orthonormal frame adapted to the surface. As in the study of curves using Frenet frames, we can read off the geometry by studying how the orthonormal frame changes along the surface. Unlike the case of curves which are only 1-dimensional, we need to examine the change of the moving frame along different tangential directions on the surface. Hence differential forms are helpful here since they require certain number of tangential vectors as input. Consider a tangential vector field Y along the surface, if we differentiate the vector field X_j along Y, then since $\{X_1, X_2, X_3\}$ is a basis, one can express $D_Y X_j$ in terms of this basis whose coefficients of course depends on the point on the surface AND the tangential vector field Y:

$$D_Y X_j = \sum_{i=1}^3 \omega_j^i(Y) X_i.$$

Recall that the directional derivative $D_Y X$ is tensorial in the Y-variable, therefore the coefficients ω_j^i in fact defines a 1-form (i.e. a $(0,1)$-tensor) on U.

Definition 4. *The connection 1-forms ω_j^i associated to an orthonormal frame $\{X_1, X_2, X_3\}$ along a surface is defined by the relation:*

$$DX_j = \sum_{i=1}^{3} \omega_j^i X_i, \qquad j = 1, 2, 3.$$

One can regard the ω_j^i as a matrix of 1-forms or a matrix-valued 1-form. We will see shortly that the components of the matrix gives information about the covariant derivatives and second fundamental form along the surface.

Lemma 5. *(i) The matrix (ω_j^i) is skew-symmetric, i.e. $\omega_j^i = -\omega_i^j$.*

(ii) For $i, j = 1, 2$, $\omega_j^i(Y) = \langle \nabla_Y X_j, X_i \rangle$, where ∇ is the covariant derivative and Y is any tangential vector field.

(iii) For $i = 1, 2$, $\omega_3^i = \iota_{X_i} h$, where h is the second fundamental form as a $(0,2)$-tensor and ι is the contraction map, i.e. $\iota_{X_i} h(Y) := h(X_i, Y)$.

Proof. For (i), we need to show that $\omega_j^i(Y) = -\omega_i^j(Y)$ for any tangential vector field Y. Since $\{X_1, X_2, X_3\}$ is an orthonormal frame *everywhere*, using the metric compatibility of D, we have

$$0 = D_Y \langle X_i, X_j \rangle = \langle D_Y X_i, X_j \rangle + \langle X_i, D_Y X_j \rangle = \omega_i^j(Y) + \omega_j^i(Y),$$

which gives the desired conclusion. (ii) is from definition since $\nabla_Y X = (D_Y X)^T$ and X_1, X_2 are tangential vectors. For (iii), since $X_3 = \nu$, using the definition of the second fundamental form h,

$$\omega_3^i(Y) = \langle D_Y \nu, X_i \rangle = h(Y, X_i) = h(X_i, Y) = \iota_{X_i} h(Y).$$

□

Therefore, we have the matrix of 1-forms

$$\omega = \begin{pmatrix} \omega_1^1 & \omega_2^1 & \omega_3^1 \\ \omega_1^2 & \omega_2^2 & \omega_3^2 \\ \omega_1^3 & \omega_2^3 & \omega_3^3 \end{pmatrix} = \begin{pmatrix} 0 & -\omega_1^2 & -\iota_{X_1} h \\ \omega_1^2 & 0 & -\iota_{X_2} h \\ \iota_{X_1} h & \iota_{X_2} h & 0 \end{pmatrix}.$$

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With these connection 1-forms ω_j^i one can write the Gauss and Codazzi equations in the following form:

Theorem 6 (Maurer-Cartan structural equations)**.** *We have for* $i, j = 1, 2, 3$*,*

$$d\omega^i_j + \sum_{k=1}^{3} \omega^i_k \wedge \omega^k_j = 0.$$

Proof. Since D is just the usual directional derivative in $\mathbb{R}^3$, we can write

$$dX_j = \sum_{i=1}^{3} \omega^i_j X_i,$$

where the left hand side means that we are taking the exterior derivative of each of the components of X_j as a vector in $\mathbb{R}^3$. Taking d on both sides again and using the fact that $d^2 = 0$, we have

$$0 = d^2 X_j = \sum_{j=1}^{3} \left(d\omega^i_j + \sum_{k=1}^{3} \omega^i_k \wedge \omega^k_j \right) X_i,$$

where we have used the Leibniz rule for differential forms $\omega \in \Omega^p(\mathbb{R}^n)$ and $\eta \in \Omega^q(\mathbb{R}^n)$,

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^p \omega \wedge d\eta.$$

□

For our convenience, we also denote the intrinsic connection 1-forms corresponding to the covariant derivative ∇ on the surface by

$$A = \begin{pmatrix} A^1_1 & A^1_2 \\ A^2_1 & A^2_2 \end{pmatrix} = \begin{pmatrix} 0 & \omega^1_2 \\ \omega^2_1 & 0 \end{pmatrix}.$$

Recall that the intrinsic Riemann curvature tensor R is defined as

$$R(X, Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

for any *tangential* vector fields X, Y, Z. Therefore, if X, Y are fixed we can regard $R(X, Y)$ as a linear operator on $T\Sigma$ defined by $Z \mapsto R(X, Y)Z$. Hence, we can define curvatures as a matrix-valued 2-forms as below:

Definition 7. *The curvature 2-forms* Ω^i_j *associated with an orthonormal frame* $\{X_1, X_2, X_3\}$ *along a surface are defined by the relation:*

$$R(X, Y)X_j = \sum_{i=1}^{2} \Omega^i_j (X, Y) X_i.$$

We have the following useful formula of the curvature 2-forms in terms of the connection 1-forms.

Theorem 8. $\Omega = dA + A \wedge A$.

Proof. Recall the Gauss equation in covariant form implies

$$\langle R(X,Y)Z, W\rangle = h(Y,Z)h(X,W) - h(X,Z)h(Y,W).$$

Therefore, using the Gauss equation and Lemma 5 (iii):

$$\langle R(X,Y)X_j, X_k\rangle = h(Y,X_j)h(X,X_k) - h(X,X_j)h(Y,X_k) = \omega_j^3(Y)\omega_k^3(X) - \omega_j^3(X)\omega_k^3(Y).$$

Since for any 1-forms ω, η, we have (Exercise: check this!)

$$(\omega \wedge \eta)(X,Y) = \omega(X)\eta(Y) - \eta(X)\omega(Y).$$

Using the definition of curvature 2-forms and Lemma 5 (i),

$$\Omega_j^k(X,Y) = (\omega_k^3 \wedge \omega_j^3)(X,Y) = -(\omega_3^k \wedge \omega_j^3)(X,Y).$$

Using the Maurer-Cartan structural equations, we have

$$\Omega_j^k(X,Y) = \left(d\omega_j^k + \sum_{i=1}^{2} \omega_i^k \wedge \omega_j^i \right)(X,Y).$$

By the definition of A, this implies $\Omega = dA + A \wedge A$. □

Finally, we use the formula above to relate our discussion to the intrinsic Gauss curvature.

Lemma 9. *$\Omega_2^1 = dA_2^1 = K\,dA$ where dA is the area 2-form of the surface.*

Proof. Recall that the Gauss curvature (or sectional curvature in general) for the 2-plane spanned by an orthonormal basis $\{X_1, X_2\}$ is given by

$$K = \langle R(X_1,X_2)X_2, X_1\rangle = \Omega_2^1(X_1,X_2).$$

This implies that $\Omega_2^1 = KdA$. The equality $\Omega_2^1 = dA_2^1$ following from Theorem 8 and that

$$A \wedge A = \begin{pmatrix} 0 & A_2^1 \\ -A_2^1 & 0 \end{pmatrix} \wedge \begin{pmatrix} 0 & A_2^1 \\ -A_2^1 & 0 \end{pmatrix} = \begin{pmatrix} -A_2^1 \wedge A_2^1 & 0 \\ 0 & -A_2^1 \wedge A_2^1 \end{pmatrix} = 0$$

since $\omega \wedge \omega = 0$ for any 1-forms ω. □

Proof of Gauss-Bonnet Theorem

We are now ready to prove the smooth local Gauss-Bonnet Theorem, which says that for any surface $\Sigma \subset \mathbb{R}^3$ with smooth boundary $\partial\Sigma$ and that Σ is diffeomorphic to a disk, we have

$$\int_\Sigma K\,dA + \int_{\partial\Sigma} k_g\,ds = 2\pi.$$

Our proof here would be very similar to the proof the Theorem of Turning Tangents which says that for any simple closed curve $\gamma \subset \mathbb{R}^2$, oriented positively, we have

$$\int_\gamma \kappa\,ds = 2\pi.$$

Note that the local Gauss-Bonnet Theorem reduces to this formula when the surface Σ is flat. Let us first recall how we proved the Theorem of Turning Tangents. Let $\{e_1, e_2\}$ be the Frenet frame of the plane curve γ. The key idea is that we can define a continuous *polar angle function* φ (up to multiples of 2π) by measuring the angle from the positive x-axis to the unit tangent vector e_1. Therefore, we have

$$e_1 = \cos\varphi \frac{\partial}{\partial x} + \sin\varphi \frac{\partial}{\partial y}, \quad e_2 = -\sin\varphi \frac{\partial}{\partial x} + \cos\varphi \frac{\partial}{\partial y}.$$

By the definition of curvature $\kappa := \langle e_1', e_2\rangle$ (we have parametrized the curve γ by arc length and $'$ means $\frac{d}{ds}$ here), we have

$$\kappa = \langle e_1', e_2\rangle = \frac{d\varphi}{ds}.$$

Integrating both sides and using a homotopy argument, one proves that

$$\int_\gamma \kappa\,ds = \int_0^L \frac{d\varphi}{ds}\,ds = \varphi(L) - \varphi(0) = 2\pi.$$

The proof of the local Gauss-Bonnet formula will be very similar to the arguments above. Except that we do not have a canonical global orthonormal frame $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\}$ for plane curves. However, we can always set up (locally) an orthonormal frame $\{X_1, X_2\}$ along the surface as before. If the boundary $\partial\Sigma$ is parametrized by arc length with $e_1 = \frac{d}{ds}$ and $e_2 = \nu \times e_1$ be the inward unit normal to $\partial\Sigma$ which is tangent to the surface Σ. Then, we can write

$$e_1 = (\cos\varphi)X_1 + (\sin\varphi)X_2, \quad e_2 = -(\sin\varphi)X_1 + (\cos\varphi)X_2.$$

In other words, e_1 makes an angle φ from X_1 measured in the intrinsic metric (g_{ij}) on the surface. Recall that the geodesic curvature $k_g := \langle \nabla_{e_1} e_1, e_2\rangle$. Using the metric compatibility and that e_1 and e_2 are orthogonal by definition, we have

$$k_g = \langle \nabla_{e_1} e_1, e_2\rangle = \frac{d}{ds}\langle e_1, e_2\rangle - \langle e_1, \nabla_{e_1} e_2\rangle = -\langle e_1, \nabla_{e_1} e_2\rangle.$$

Using the expression of e_1, e_2 in terms of X_1, X_2, we get

$$\nabla_{e_1} e_2 = -\frac{d\varphi}{ds} e_1 - (\sin\varphi)\nabla_{e_1} X_1 + (\cos\varphi)\nabla_{e_1} X_2.$$

Using the definition of the connection forms A^i_j and that $A^1_2 = -A^2_1$, we have

$$-\langle e_1, \nabla_{e_1} e_2 \rangle = \frac{d\varphi}{ds} - A^1_2(e_1).$$

Therefore, putting all these together, we obtain

$$\frac{d\varphi}{ds} = k_g + A^1_2(e_1).$$

Integrating on both side along $\partial\Sigma$, using Stokes' Theorem and Lemma 9, we have

$$2\pi = \int_0^L \frac{d\varphi}{ds}\, ds = \int_{\partial\Sigma} k_g\, ds + \int_{\partial\Sigma} A^1_2 = \int_{\partial\Sigma} k_g\, ds + \int_\Sigma dA^1_2 = \int_{\partial\Sigma} k_g\, ds + \int_\Sigma K\, dA.$$

This proves the Gauss-Bonnet theorem as wished. (Note that the integral on the left hand side is 2π because we can define the polar angle φ_t with respect to the interpolating metric $g_t = (1-t)\delta + tg$ where δ is the Euclidean metric on U and g is the first fundamental form of Σ. Since the integral is always an integer multiple of 2π and that it should depend continuously with respect to t. As the integral is 2π for the Euclidean metric g_0 by the Theorem of Turning Tangents, it is also true for $g_1 = g$ by this continuity argument.)