MATH 4030 Differential Geometry
Lecture Notes Part 1
last revised on September 28, 2015

The subject "*Differential Geometry*" can be vaguely defined as the area of mathematics which uses differential and integral calculus to study geometry. During the last century the field has expanded enormously into an intertwining subject between analysis, complex and algebraic geometry, representation theory and mathematical physics etc. What we are going to cover in this course is the classical differential geometry of curves and surfaces which dates back to the times of Euler and Gauss. We hope this course can serve as an introduction to the field which opens a lot more doors into the fascinating world of geometry.

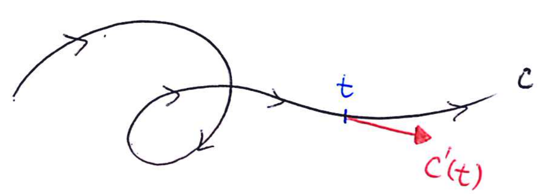
We will mainly follow the textbook written by Wolfgang Kühnel but additional materials would be added from other reference where appropriate. Students are advised to read the textbook with care as there are some typos in the text.

Regular parametrized curves in $\mathbb{R}^n$

We will be studying curves, surfaces (and submanifolds in general) in the n-dimensional Euclidean space $\mathbb{R}^n$. We think of $\mathbb{R}^n := \{x = (x^1, \cdots, x^n) : x^i \in \mathbb{R}\}$ as an n-dimensional vector space over $\mathbb{R}$ which is equipped with the standard inner product $\langle x, y\rangle := \sum_{i=1}^n x^i y^i$ and the norm $\|x\| := \left(\sum_{i=1}^n (x^i)^2\right)^{1/2}$. (In Riemannian geometry, one consider a differentiable manifold M^n equipped with an inner product g_p on each tangent space T_pM which varies smoothly in p. This notion is essential in Einstein's general theory of relativity where the spacetime is described by a 4-dimensional Lorentzian manifold.)

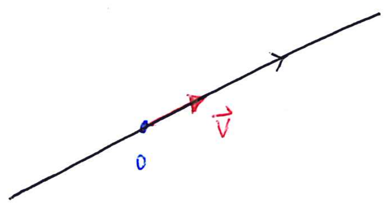
Just like we learn calculus by first understanding single variable calculus, we will begin with the simplest object - *curves* - a 1-dimensional object.

Definition 1. *A (smooth) parametrized curve in $\mathbb{R}^n$ is a smooth map $c : [a, b] \to \mathbb{R}^n$ given in coordinates by $t \mapsto (c_1(t), \cdots, c_n(t))$. The curve c is said to be regular if $c'(t) \neq 0$ for all $t \in (a, b)$.*



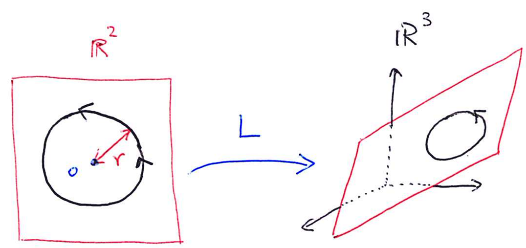
One can think of the parametrized curve $c(t)$ as the trajectory of an object moving in $\mathbb{R}^n$ and $c(t)$ is its position at time t. The condition $c'(t) \neq 0$ then just means that the object does not stop moving at any particular time. Let us first look at some basic examples.

1. *Straight line in* $\mathbb{R}^n$: given a vector $\vec{v} \in \mathbb{R}^n$, $c(t) = t\vec{v}$ for $t \in (-\infty, \infty)$ is parametrizing the straight line through the origin parallel to the vector $\vec{v}$.



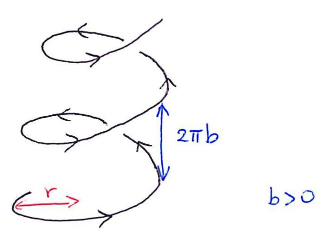
Exercise: Give a parametrization of the line segment from a point p to another point q in $\mathbb{R}^n$.

2. *Circles*: the parametrized curve $c(\theta) = r(\cos\theta, \sin\theta)$, $\theta \in [0, 2\pi]$, is a circle of radius r centered at the origin. If $L : \mathbb{R}^2 \to \mathbb{R}^n$ is an affine linear map, then $L \circ c$ is an ellipse in $\mathbb{R}^n$.



Exercise: Can you give a parametrization of the unit circle centered at origin so that $\|c'(t)\| = 1$ for all t?

3. *Helix in* $\mathbb{R}^3$: $c(t) = (r\cos t, r\sin t, bt)$, $t \in \mathbb{R}$.



4. *Neil parabola*: consider the parametrized curve $c(t) = (t^2, t^3)$, $t \in \mathbb{R}$, which lies on the set $\{(x, y) \in \mathbb{R}^2 : x^3 = y^2\}$. Note that even though the coordinate functions of $c(t)$ are smooth,

the image curve is not smooth at the origin. In fact, the curve $c(t)$ fails to be regular at $t = 0$. *Exercise: Show that there is NO regular parametrization of the Neil parabola.*

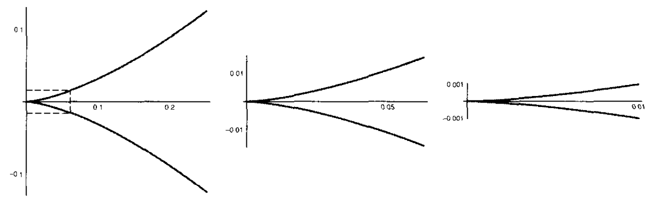


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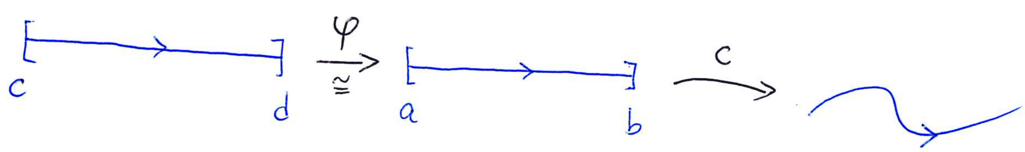
Figure 2.2. Neil parabola

Arc length parametrization

From Definition **??** we see that a *regular parametrized curve* is not just the image curve as a "stationary" object but it also includes the parametrization - how the image curve is being traced out. Sometimes we would like to think of the curve as a stationary object, forgetting about the parametrization. This concept can be made rigorous by the following definitions.

Definition 2. *Let $c : [a, b] \to \mathbb{R}^n$ be a parametrized curve. If $\varphi : [c, d] \to [a, b]$, $\varphi' > 0$, is an oriented (smooth) diffeomorphism, then $c \circ \varphi : [c, d] \to \mathbb{R}^n$ is said to be a reparametrization of c.*



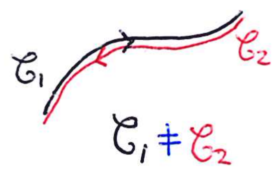
Definition 3. *Let $c_1 : [a, b] \to \mathbb{R}^n$ and $c_2 : [c, d] \to \mathbb{R}^n$ be two parametrized curves. We say that $c_1 \sim c_2$ if c_2 is a reparametrization of c_1.*

Lemma 4. *$\sim$ is an equivalence relation on the space of regular parametrized curves in $\mathbb{R}^n$.*

Proof. Exercise. (Note that if c_1 is regular and $c_1 \sim c_2$, then c_2 is also regular.) □

Definition 5. *A regular curve, $\mathcal{C} = [c]$, is an equivalence class of regular parametrized curves.*

Note that a regular curve comes with a fixed orientation.



Now, we study an important geometric invariant of a regular curve. We will see later that this is the only intrinsic geometric invariant.

Definition 6. *The length of a regular curve $\mathcal{C} = [c]$ is defined to be*

$$L(\mathcal{C}) := \int_a^b \|c'(t)\| \, dt \tag{1}$$

where $c : [a, b] \to \mathbb{R}^n$ is a regular parametrized curve in the class $\mathcal{C}$.

Lemma 7. *The notion of length, $L(\mathcal{C})$, is well-defined, i.e. the right hand side of (**??**) is independent of the choice of $c \in \mathcal{C}$.*

Proof. This is basically a consequence of the chain rule. Suppose $\varphi : [c, d] \to [a, b]$ is an oriented diffeomorphism with $\varphi'(\tau) > 0$ for all $\tau \in (c, d)$ and $\varphi(c) = a$, $\varphi(d) = b$. Then, $\tilde{c} := c \circ \varphi : [c, d] \to \mathbb{R}^n$ is a reparametrization of c so $[\tilde{c}] = \mathcal{C}$. By the chain rule,

$$\frac{d}{d\tau}\tilde{c}(\tau) = c'(\varphi(\tau)) \cdot \varphi'(\tau).$$

Therefore, the change of variable formula gives (we use $\varphi'(\tau) > 0$ here)

$$\int_c^d \|\tilde{c}'(\tau)\| d\tau = \int_c^d \|c'(\varphi(\tau)\|\varphi'(\tau) d\tau = \int_a^b \|c'(t)\| dt.$$

Hence, the right hand side of (**??**) is independent of reparametrization and thus $L(\mathcal{C})$ is well-defined. □

Since a regular curve $\mathcal{C}$ is the image (oriented) curve with all its possible parametrizations, a natural question is whether there is any *special* parametrization which gives a canonical representation c for any regular curve $\mathcal{C}$. The answer is YES and this is given by the so-called *arc-length parametrization*.

Definition 8. *A regular parametrized curve $c(t) : [a, b] \to \mathbb{R}^n$ is said to be parametrized by arc length if $\|c'(t)\| = 1$ for all $t \in (a, b)$. In this case, we will usually denote t by the variable s, called the arc length parameter.*

The following lemma below says that any regular parametrized curve can be reparametrized by arc length. This implies that there is no intrinsic local invariant for a curve, and the only global invariant is given by the total arc length $L(\mathcal{C})$.

Lemma 9. *For any regular curve $\mathcal{C}$, there exists an arc-length parametrization $c \in \mathcal{C}$. Moreover, the arc-length parametrization is unique up to translations in the domain interval.*

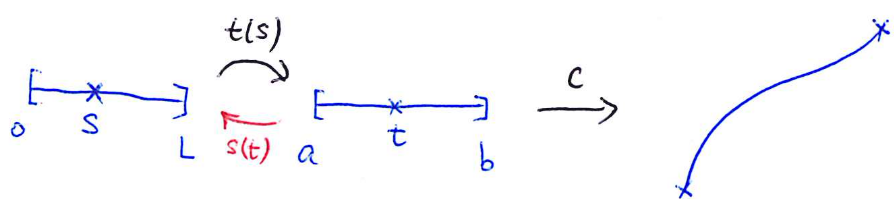
Proof. Take any regular parametrization $c(t) : [a, b] \to \mathbb{R}^n$ for $\mathcal{C}$. Define the *arc-length* function $s : [a, b] \to [0, L]$, where $L = L(\mathcal{C})$, by

$$s(t) := \int_a^t \|c'(\tau)\| \, d\tau.$$

By the fundamental theorem of calculus, $s'(t) = \|c'(t)\| > 0$ for all $t \in (a, b)$. (Note that c is regular.) Therefore, the inverse function theorem says that there is a smooth inverse $t = t(s) : [0, L] \to [a, b]$ to the function $s = s(t) : [a, b] \to [0, L]$. Consider the reparametrization of c given by $\tilde{c}(s) := c(t(s)) : [0, L] \to \mathbb{R}^n$, then by the chain rule,

$$\frac{d}{ds}\tilde{c}(s) = c'(t(s))\frac{dt}{ds} = c'(t(s))\left(\frac{ds}{dt}\right)^{-1} = \frac{c'(t(s))}{\|c'(t(s))\|},$$

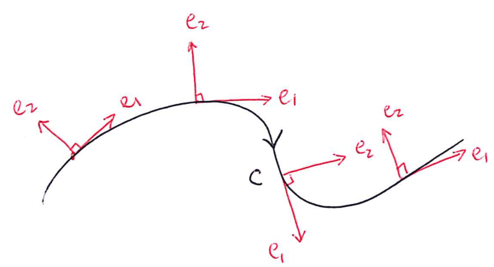
which has unit length. Therefore, $\tilde{c}$ is an arc-length parametrization of $\mathcal{C}$. We leave the proof of the uniqueness as an exercise. □



As an exercise, try to give the arc length parametrization of all the examples in the first section.

The method of moving frame

The idea of moving frames is a very useful tool in differential geometry which was promoted by the great geometers Cartan and S.S. Chern in the 20th century. The standard coordinate basis of $\mathbb{R}^n$ provides a *parallel* global inertial frame. However, it is not always the most convenient frame in practice. For example, image a car driving on a curvy road, the concept of "driving forward along the road" makes sense to the driver (as long as the road is well-paved and there is no branching). However, the *forward* direction depends on your position on the road and the forward direction turns more if the road is more curvy. This intuition tells us that the *curvature* of a curve is measured by the rate of the turning of the *forward* direction. Mathematically, the *forward* and *sideways* direction along a curve is given by a *frame*, i.e. an orthonormal basis, which changes from point to point as one moves along the curve. This explains the term "*moving frame*".



We now make a general definition and then we will specialize to the case of two and three dimensions.

Definition 10. *Let $c(s)$ be a regular curve in $\mathbb{R}^n$ parametrized by arc length. We say that c is a* <u>*Frenet curve*</u> *if the set of $n-1$ vectors $\{c'(s), c''(s), \cdots, c^{(n-1)}(s)\}$ in $\mathbb{R}^n$ are linearly independent for all s. For a Frenet curve $c(s)$, the* <u>*Frenet frame associated to c*</u> *is the set of n vectors $\{e_1(s), \cdots, e_n(s)\}$ in $\mathbb{R}^n$ such that for all s,*

(i) $\{e_1(s), \cdots, e_n(s)\}$ *is a positively oriented orthonormal basis of $\mathbb{R}^n$, i.e.*

$$det(e_1(s), e_2(s), \cdots, e_n(s)) = 1;$$

(ii) $span\{e_1(s), \cdots, e_k(s)\} = span\{c'(s), \cdots, c^{(k)}(s)\}$ *for all $k = 1, 2, \cdots, n$;*

(iii) $\langle c^{(k)}(s), e_k(s)\rangle > 0$ *for all $k = 1, 2, \cdots, n$.*

Lemma 11. *Given a Frenet curve c, there exists a unique Frenet frame associated to c.*

Proof. The proof is just the Gram-Schmidt orthogonalization process applied to the linear independent set of vectors $\{c'(s), c''(s), \cdots, c^{(n-1)}(s)\}$ to get an orthonormal set $\{e_1(s), \cdots, e_{n-1}(s)\}$. The last vector $e_n(s)$ is then obtained using the orientation and the metric. (Note: It is not defined by $c^{(n)}(s)$, which might be zero for a Frenet curve. Can you give an example?) $\square$

The moving frame $s \mapsto \{e_1(s), \cdots, e_n(s)\}$ is just a section of the *frame bundle* along the curve c. The main reason of introducing the moving frame is that it greatly simplifies the equation of motions. We will now focus on the case $n = 2$ or 3, and we will return to the general case later.

Frenet equations for $n = 2$ and 3

First, we study *plane curves*, i.e. curves in $\mathbb{R}^2$.

Proposition 12. *(1) Every regular curve parametrized by arc length is a Frenet curve.*

(2) Let $c(s)$ be a regular curve parametrized by arc length. Then the Frenet frame associated to c is given by

$$e_1(s) := c'(s), \quad e_2(s) := Je_1(s),$$

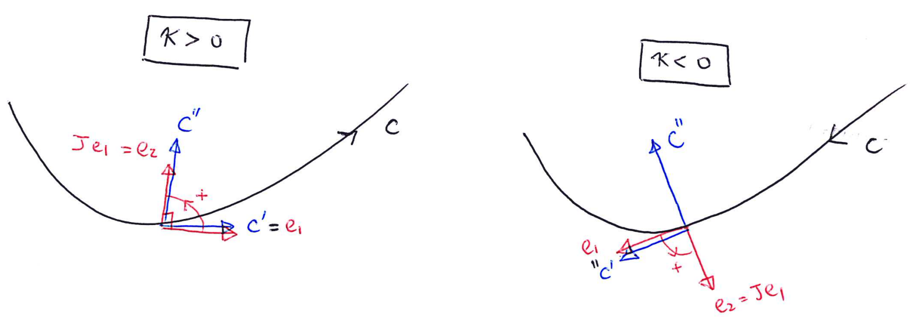
where $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is the counterclockwise rotation by $\pi/2$. Moreover, $c''(s) = \kappa(s)e_2(s)$ for all s and the function $\kappa(s) \in \mathbb{R}$ is called the (signed) curvature of the curve c.

Proof. When $n = 2$, an arc-length parametrized curve $c(s)$ is Frenet if and only if $c'(s) \neq 0$ for all s, which holds automatically since $\|c'(s)\| = 1$ by the definition of arc length parametrization. This proves (1). For (2), notice that $\langle c'(s), c'(s)\rangle \equiv 1$ for all s. Differentiating in s gives

$$2\langle c'(s), c''(s)\rangle = \frac{d}{ds}\langle c'(s), c'(s)\rangle = 0.$$

Since $e_1(s) = c'(s)$, we must have $c''(s)$ parallel to $e_2(s)$. This proves (2). □

Note that reversing the orientation of the curve flips the sign of the curvature (Exercise: Prove this!)



Remark: The notion of curvature κ is NOT an *intrinsic* property of the curve itself! In other words, an ant walking on the curve whose world is just the 1-dimensional curve cannot detect κ. The "*centrifugal force*" that one feels when traveling on a circle is always *perpendicular* to the curve, hence is something the ant cannot feel if its world is only 1-dimensional along the curve. Therefore, the curvature κ is an *extrinsic* property that tells us how the 1-dimensional curve is sitting inside the 2-dimensional plane. The concept of intrinsic and extrinsic properties is an intriguing concept in differential geometry which would re-appear from time to time throughout this course.

As stated at the end of the previous section, the main reason for introducing the Frenet frame is that the equation of motions simplifies substantially in this adapted frame (This is similar in spirit to choosing a good coordinate system for a specific problem - however - there is some subtle differences between a frame and a coordinate system, we will address this point later in the course).

Proposition 13. *The Frenet frame $\{e_1(s), e_2(s)\}$ of a Frenet curve in $\mathbb{R}^2$ satisfies the following system of linear ODEs called the* <u>*Frenet equations*</u>*:*

$$\begin{pmatrix} e_1(s) \\ e_2(s) \end{pmatrix}' = \begin{pmatrix} 0 & \kappa(s) \\ -\kappa(s) & 0 \end{pmatrix} \begin{pmatrix} e_1(s) \\ e_2(s) \end{pmatrix},$$

where $\kappa(s)$ is the (signed) curvature of the curve (defined in Proposition **??** *(2)).*

Proof. By definition of κ and e_1, we have $e_1'(s) = c''(s) = \kappa(s)e_2(s)$. On the other hand, since $\langle e_2(s), e_2(s)\rangle \equiv 1$, we have $\langle e_2'(s), e_2(s)\rangle = 0$ and thus $e_2'(s)$ is parallel to $e_1(s)$. Moreover,

$$\langle e_2'(s), e_1(s)\rangle = \langle e_2(s), e_1(s)\rangle' - \langle e_2(s), e_1'(s)\rangle = -\kappa(s).$$

Therefore, $e_2'(s) = -\kappa(s)e_1(s)$ and we have proved the proposition. □

Note that in the proof above, we have repeatedly used the fact that the metric $\langle \, , \rangle$ is constant and therefore do not need to be differentiated. In Riemannian geometry, one needs the metric g to be *parallel* in the sense that $\nabla g = 0$ where ∇ is a canonical *connection* determined by g.

Now, we go up one dimension to study *space curves*, i.e. curves in $\mathbb{R}^3$.

Proposition 14. *Let $c(s)$ be a regular curve in $\mathbb{R}^3$ parametrized by arc length.*

(1) c is Frenet if and only if $c''(s) \neq 0$ for all s.

(2) If c is Frenet, then the Frenet frame associated to c is given by

$$e_1(s) = c'(s), \quad e_2(s) = \frac{c''(s)}{\|c''(s)\|}, \quad e_3(s) = e_1(s) \times e_2(s).$$

Classically, e_1, e_2, e_3 are called the <u>*tangent*</u>*,* <u>*principal normal*</u> *and* <u>*binormal vectors*</u> *respectively.*

Proof. Since $c(s)$ is parametrized by arc length, $\langle c'(s), c'(s)\rangle \equiv 1$ implies that $\langle c'(s), c''(s)\rangle \equiv 0$. Therefore, if $c''(s) \neq 0$, then $\{c'(s), c''(s)\}$ is linearly independent. This proves (1). The statement (2) then follows from (1) and the definition of cross product in $\mathbb{R}^3$. □

Definition 15. *Given a Frenet curve in $\mathbb{R}^3$ with associated Frenet frame $\{e_1(s), e_2(s), e_3(s)\}$, we define the* <u>*curvature*</u> *and* <u>*torsion*</u> *of c respectively as:*

$$\kappa(s) := \|c''(s)\| \qquad \textit{and} \qquad \tau(s) := \langle e_2'(s), e_3(s)\rangle.$$

Note that in contrast with plane curves, the curvature is always positive $\kappa > 0$ by definition, while the torsion τ can have a sign. Geometrically, the curvature κ measures how the tangent vector turns and the torsion τ measures how the plane containing c' and c'' turns. We now derive the Frenet equations for a space curve.

Proposition 16. *The Frenet frame $\{e_1(s), e_2(s), e_3(s)\}$ of a Frenet curve in $\mathbb{R}^3$ satisfies the following Frenet equations:*

$$\begin{pmatrix} e_1(s) \\ e_2(s) \\ e_3(s) \end{pmatrix}' = \begin{pmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{pmatrix} \begin{pmatrix} e_1(s) \\ e_2(s) \\ e_3(s) \end{pmatrix},$$

where $\kappa(s)$ and $\tau(s)$ is the curvature and torsion of the curve.

Proof. For simplicity, we will omit the dependence of s in the proof. By definitions, we have $e_1' = \|e_1'\| e_2 = \kappa e_2$. Moreover, since e_2 is parallel to e_1', we have

$$e_3' = e_1' \times e_2 + e_1 \times e_2' = e_1 \times e_2'.$$

Therefore, $\langle e_3', e_1 \rangle = 0$. Moreover, we have $\langle e_3', e_3 \rangle = 0$ (as $\|e_3\| \equiv 1$) and $\langle e_3', e_2 \rangle = -\langle e_3, e_2' \rangle = -\tau$ (as $\langle e_2, e_3 \rangle \equiv 0$). As a result, we obtain the last equation $e_3' = -\tau e_2$. For the second equation, by a similar argument we have $\langle e_2', e_2 \rangle = 0$ and

$$\langle e_2', e_1 \rangle = -\langle e_2, e_1' \rangle = -\kappa,$$

$$\langle e_2', e_3 \rangle = -\langle e_2, e_3' \rangle = \tau.$$

Therefore, $e_2' = -\kappa e_1 + \tau e_3$, which proves the proposition. □

Plane curves

From the Frenet equations of a plane curve, we see that there is only one (extrinsic) geometric invariant given by the signed curvature κ. Geometrically, it measures quantitatively how fast is the unit tangent vector $e_1 = c'$ turning *towards* the e_2-direction. Let us look at two basic examples:

1. *Straight line*: Let $c(s) = s\vec{v}$, $s \in \mathbb{R}$ be the arc length parametrization of the line parallel to the unit vector $\vec{v} \in \mathbb{R}^2$. Then $c''(s) = 0$ and hence $\kappa(s) \equiv 0$.

2. *Circles*: Consider an arc length parametrization of the circle of radius r centered at p:

$$c(s) = p + r\left(\cos\frac{s}{r}, \sin\frac{s}{r}\right), \qquad s \in \mathbb{R}.$$

   Then the Frenet frame is given by

$$e_1(s) = c'(s) = \left(-\sin\frac{s}{r}, \cos\frac{s}{r}\right), \qquad \text{and} \qquad e_2(s) = Je_1(s) = -\left(\cos\frac{s}{r}, \sin\frac{s}{r}\right).$$

   Therefore, the curvature is given by $\kappa(s) = \langle c''(s), e_2(s) \rangle \equiv 1/r$. (Note that if we parametrized the circle in the clockwise direction, we get $\kappa \equiv -1/r$ instead.)

In fact, these are the only examples of plane curves with constant curvature.

Proposition 17. *The lines and circles are the only plane curves with constant curvature κ.*

The proposition above follows easily as a corollary of the following general theorem.

Theorem 18. *The curvature $\kappa(s)$ determines the curve $c(s)$ uniquely up to translations and rotations (i.e. rigid motions of $\mathbb{R}^2$).*

Proof. By the existence and uniqueness of the solution to linear ODE systems, the Frenet equations is uniquely solvable with any given initial data:

$$\begin{cases} \begin{pmatrix} e_1(s) \\ e_2(s) \end{pmatrix}' = \begin{pmatrix} 0 & \kappa(s) \\ -\kappa(s) & 0 \end{pmatrix} \begin{pmatrix} e_1(s) \\ e_2(s) \end{pmatrix} \\ \\ e_1(0) = \vec{v}, \quad e_2(0) = J\vec{v}. \end{cases}$$

where $\vec{v} \in \mathbb{R}^2$ is a unit vector. Then, for any fixed $p \in \mathbb{R}^2$, the initial value problem below is uniquely solvable for $c(s)$:

$$\begin{cases} c'(s) = e_1(s), \\ c(0) = p. \end{cases}$$

Therefore, given any initial data $(p, \vec{v})$, the curve $c(s)$ is uniquely determined by the curvature $\kappa(s)$. □

Recall that the tangent line to a curve at a point p is the linear approximation of the curve near p. We say that the tangent line has *contact order* 1 *with the curve at* p. If we want to approximate the curve near p with something of contact order 2, it leads to the notion of *osculating circle*.

Definition 19. *If $c(s)$ is an arc-length parametrized plane curve with $\kappa(s_0) \neq 0$ for some s_0. Then, the osculating circle of c at s_0 is the unique circle of radius $1/\kappa(s_0)$ centered at $\alpha(s_0) := c(s_0) + \frac{1}{\kappa(s_0)} e_2(s_0)$.*

Proposition 20. *The osculating circle of c at s_0 has contact order 2 with c at $s = s_0$, i.e.*

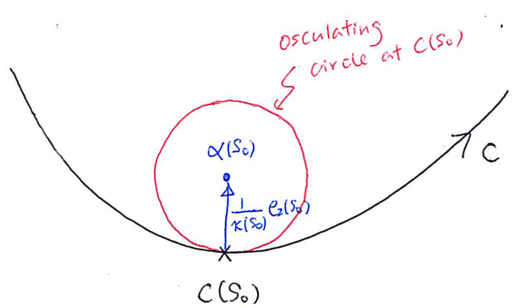
$$\left. \frac{d^{(k)}}{ds^{(k)}} \right|_{s=s_0} \|c(s) - \alpha(s_0)\| = 0, \qquad \textit{for } k = 1, 2.$$

Proof. Note that $\|c(s) - \alpha(s_0)\| > 0$ for $s \sim s_0$ so the function above is smooth near s_0. Moreover,

$$\left. \frac{d}{ds} \right|_{s=s_0} \|c(s) - \alpha(s_0)\|^2 = \langle c'(s_0), c(s_0) - \alpha(s_0) \rangle = \langle e_1(s_0), -\frac{1}{\kappa(s_0)} e_2(s_0) \rangle = 0.$$

$$\begin{aligned} \left. \tfrac{d^2}{ds^2} \right|_{s=s_0} \|c(s) - \alpha(s_0)\|^2 &= \langle c''(s_0), c(s_0) - \alpha(s_0) \rangle + \langle c'(s_0), c'(s_0) \rangle \\ &= \langle \kappa(s_0) e_2(s_0), -\tfrac{1}{\kappa(s_0)} e_2(s_0) \rangle + \langle e_1(s_0), e_1(s_0) \rangle \\ &= (-1) + 1 = 0. \end{aligned}$$

□



We can do a Taylor expansion of a plane curve $c(s)$ near $s = 0$:

$$c(s) = c(0) + sc'(0) + \frac{s^2}{2}c''(0) + \frac{s^3}{6}c'''(0) + O(s^4).$$

Assuming without loss of generality that $c(0) = (0,0)$ and $c'(0) = (1,0)$ (hence the Frenet frame at origin is $e_1(0) = (1,0)$ and $e_2(0) = (0,1)$), then using the Frenet equations we have

$$c''(0) = e_1'(0) = \kappa(0)e_2(0),$$

$$c'''(0) = \kappa'(0)e_2(0) + \kappa(0)e_2'(0) = -\kappa(0)^2 e_1(0) + \kappa'(0)e_2(0).$$

Putting these back to the Taylor expansion formula, we get

$$c(s) = \left(s - \frac{\kappa(0)^2}{6}s^3, \frac{\kappa(0)}{2}s^2 + \frac{\kappa'(0)}{6}s^3\right) + O(s^4).$$

This also illustrates why the curvature function $\kappa(s)$ (and its higher derivatives) alone is enough to determine the curve $c(s)$ completely up to rigid motions.

Space curves

From the Frenet equations for space curve, there are two geometric quantities κ and τ that describe the change of the Frenet frame along the curve. The curvature κ measures the rate of change of the unit tangent vector (which is positive by definition, in contrast with the case for plane curve), while the torsion τ measures how much the curve is deviated from being a plane curve. The example below helps illustrate these geometric implications.

Consider a helix with arc length parametrization

$$c(s) = \left(r\cos\frac{s}{\sqrt{r^2+b^2}}, r\sin\frac{s}{\sqrt{r^2+b^2}}, \frac{bs}{\sqrt{r^2+b^2}}\right),$$

differentiating in s gives

$$e_1(s) = c'(s) = \frac{1}{\sqrt{r^2+b^2}}\left(-r\sin\frac{s}{\sqrt{r^2+b^2}}, r\cos\frac{s}{\sqrt{r^2+b^2}}, b\right),$$

$$c''(s) = \frac{r}{r^2+b^2}\left(-\cos\frac{s}{\sqrt{r^2+b^2}}, -\sin\frac{s}{\sqrt{r^2+b^2}}, 0\right).$$

Therefore, the curvature is $\kappa(s) = \|c''(s)\| = \frac{r}{r^2+b^2}$ and

$$e_2(s) = \frac{c''(s)}{\|c''(s)\|} = \left(-\cos\frac{s}{\sqrt{r^2+b^2}}, -\sin\frac{s}{\sqrt{r^2+b^2}}, 0\right).$$

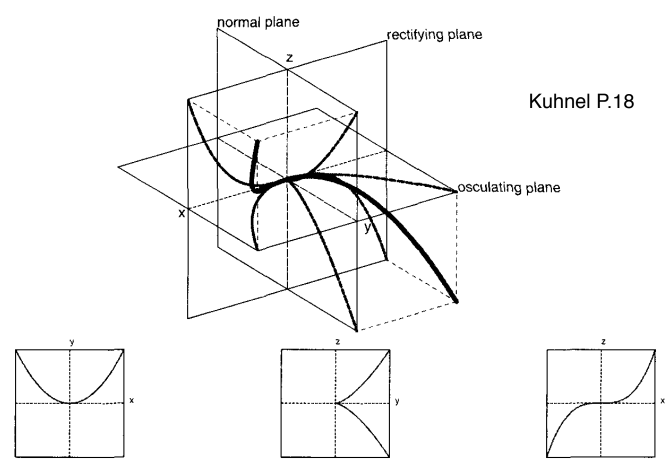
Therefore,

$$e_3(s) = e_1(s)\times e_2(s) = \frac{1}{\sqrt{r^2+b^2}}\left(b\sin\frac{s}{\sqrt{r^2+b^2}}, -b\cos\frac{s}{\sqrt{r^2+b^2}}, r\right).$$

Hence, the torsion is

$$\tau(s) = -\langle e_3'(s), e_2(s)\rangle = \frac{b}{r^2+b^2}.$$

As for the case of plane curve, we can do a Taylor expansion of a Frenet curve in $\mathbb{R}^3$ to study the local behavior of the curve (see Tutorial Notes 1):



(a) Osculating plane

(b) Normal plane

(c) Rectifying plane

Figure 2.5. Three projections of the space curve $xe_1 + ye_2 + ze_3$

Next, we introduce the Darboux rotation vector $\vec{D}$ in $\mathbb{R}^3$. Recall from the Frenet equations that the 3×3 coefficient matrix A, sometimes called the Frenet matrix, is skew-symmetric, i.e. $A^T = -A$. Therefore, $A \in so(3)$ is an infinitesimal rotation in $\mathbb{R}^3$ and is therefore given by $\vec{D}\times(\cdot)$ for some vector

$\vec{D}$ called the Darboux vector (which depends on s since A depends on s). We digress a little bit to see why skew-symmetric matrices correspond to infinitesimal rotations (in fact in any dimension, $n = 3$ is only used to conclude that the infinitesimal rotation is given by the cross product with a vector $\vec{D}$).

Let $SO(n)$ be the special orthogonal group of $\mathbb{R}^n$ which is the group of orientation-preserving isometries of $\mathbb{R}^n$:

$$SO(n) := \{A \in GL(n) : AA^T = A^T A = I, \det A = 1\}.$$

This is a classical example of a Lie group, which is a differentiable manifold with a group structure. Therefore, we can look at the tangent space at the identity element $I \in SO(n)$, which is called the Lie algebra of the Lie group $SO(n)$, denoted by $so(n)$. To see that $so(n)$ consists of skew-symmetric matrices, suppose we have a smooth family $A(t) \in SO(n)$, $t \in (-\epsilon, \epsilon)$, such that $A(0) = I$. Since $A(t) \in SO(n)$ for all t, we have the identity

$$A(t)A^T(t) \equiv I.$$

Differentiating at $t = 0$ and using the initial condition $A(0) = A^T(0) = I$, we obtain

$$A'(0) + (A')^T(0) = O,$$

i.e. $(A')^T(0) = -A'(0)$. In other words, any tangent vector to $SO(n)$ at the identity I is given by a skew-symmetric matrix $A'(0)$, therefore,

$$so(n) = \{B \in gl(n) : B^T = -B\},$$

where $gl(n)$ is the set of all $n \times n$ matrices, the Lie algebra of the Lie group $GL(n)$ of all $n \times n$ invertible matrices.

Now, we go back to find the *Darboux vector* $\vec{D}$ in terms of the Frenet frame $\{e_1, e_2, e_3\}$. Suppose $\vec{D} = ae_1 + be_2 + ce_3$, then using the definition of $\vec{D}$ and the Frenet equations, we have

$$\begin{pmatrix} \vec{D} \times e_1 \\ \vec{D} \times e_2 \\ \vec{D} \times e_3 \end{pmatrix} = \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}' = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}.$$

Using that $\{e_1, e_2, e_3\}$ is an oriented orthonormal frame and comparing coefficients, we get

$$\vec{D} = \tau e_1 + \kappa e_2.$$

Note that $\|\vec{D}\| = \sqrt{\tau^2 + \kappa^2}$ represents the speed of rotation of the Frenet frame. (As an exercise, show that a helix has constant Darboux vector which is parallel to the "axis" of the helix.)

Recall that the only plane curves with constant curvature are circles. Similarly, in the case of space curves, we want to characterize those curves in $\mathbb{R}^3$ which lies on a sphere. Let

$$\mathbb{S}^2_R(p) := \{x \in \mathbb{R}^3 : \|x - p\| = R\}$$

be the sphere of radius $R > 0$ centered at $p \in \mathbb{R}^3$.

Theorem 21. *Let $c(s)$ be a Frenet curve in $\mathbb{R}^3$ parametrized by arc length. Suppose that $\tau \neq 0$ everywhere. Then c lies on a sphere if and only if the following equation holds:*

$$\frac{\tau}{\kappa} = \left(\frac{\kappa'}{\kappa^2\tau}\right)'. \tag{2}$$

Proof. Suppose we have an arc-length parametrized Frenet curve $c(s)$ lying on a sphere $\mathbb{S}^2_R(p)$. We will prove the conclusion assuming $p = 0$ (Exercise: Write the proof for an arbitrary p). By hypothesis, $\langle c, c\rangle \equiv R^2 = \text{constant}$. Differentiating with respect to s, using $e_1 := c'$,

$$0 = \frac{1}{2}\langle c, c\rangle' = \langle c', c\rangle = \langle e_1, c\rangle.$$

Differentiating again and using the Frenet equation $e_1' = \kappa e_2$,

$$0 = \langle e_1, c\rangle' = \langle e_1', c\rangle + \langle e_1, c'\rangle = \langle \kappa e_2, c\rangle + \langle e_1, e_1\rangle,$$

which implies that $\langle e_2, c\rangle = -1/\kappa$. Differentiate once more and using Frenet equation $e_2' = -\kappa e_1 + \tau e_3$, we have

$$\langle -\kappa e_1 + \tau e_3, c\rangle + \langle e_2, e_1\rangle = \langle e_2', c\rangle + \langle e_2, c'\rangle = \langle e_2, c\rangle' = \frac{\kappa'}{\kappa^2},$$

which implies that $\langle e_3, c\rangle = \frac{\kappa'}{\kappa^2\tau}$ (recall that $\langle e_1, c\rangle = 0$). Combining all these results, we have

$$c = -\frac{1}{\kappa}e_2 + \frac{\kappa'}{\kappa^2\tau}e_3.$$

In other words, for all s,

$$c + \frac{1}{\kappa}e_2 - \frac{\kappa'}{\kappa^2\tau}e_3 = 0.$$

Differentiating one last time and using the Frenet equations,

$$\begin{aligned} 0 &= c' - \tfrac{\kappa'}{\kappa^2}e_2 + \tfrac{1}{\kappa}e_2' - \left(\tfrac{\kappa'}{\kappa^2\tau}\right)' e_3 - \tfrac{\kappa'}{\kappa^2\tau}e_3' \\ &= e_1 - \tfrac{\kappa'}{\kappa^2}e_2 + \tfrac{1}{\kappa}(-\kappa e_1 + \tau e_3) - \left(\tfrac{\kappa'}{\kappa^2\tau}\right)' e_3 - \tfrac{\kappa'}{\kappa^2\tau}(-\tau e_2) \\ &= \left(\tfrac{\tau}{\kappa} - \left(\tfrac{\kappa'}{\kappa^2\tau}\right)'\right) e_3, \end{aligned}$$

which implies the desired conclusion (**??**).

Suppose, on the other hand, (**??**) holds. Reversing the argument above implies that

$$c + \frac{1}{\kappa}e_2 - \frac{\kappa'}{\kappa^2\tau}e_3 \equiv p,$$

for some constant $p \in \mathbb{R}^3$. We claim that c lies on a sphere $\mathbb{S}^2_R(p)$ centered at p for some radius $R > 0$. It is equivalent to showing that $\|c - p\|^2 \equiv \text{constant}$, which follows from

$$\frac{1}{2}\langle c-p, c-p\rangle' = \langle c', c-p\rangle = \langle e_1, -\frac{1}{\kappa}e_2 + \frac{\kappa'}{\kappa^2\tau}e_3\rangle \equiv 0.$$

□

In general, if a Frenet curve $c(s)$ does not lie on a sphere, we can still define an approximating sphere which the curve roughly lies on.

Definition 22. *For a Frenet curve $c(s)$ in $\mathbb{R}^3$ such that $\tau(s_0) \neq 0$, the osculating sphere of c at $s = s_0$ is the the sphere $\mathbb{S}^2_R(p)$ where*

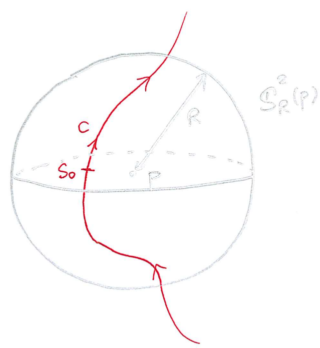
$$p = c(s_0) + \frac{1}{\kappa(s_0)} e_2(s_0) - \frac{\kappa'(s_0)}{\kappa^2(s_0)\tau(s_0)} e_3(s_0),$$

$$R = \|c(s_0) - p\| = \left(\frac{1}{\kappa^2(s_0)} + \left(\frac{\kappa'(s_0)}{\kappa^2(s_0)\tau(s_0)} \right)^2 \right)^{\frac{1}{2}}.$$

Proposition 23. *The Frenet curve c and the osculating sphere S at $c(s_0)$ intersect with contact order 3, i.e.*

$$\left.\frac{d^{(i)}}{ds^{(i)}}\right|_{s=s_0} \|c(s) - p\| = 0, \qquad \text{for } i = 1, 2, 3.$$

Proof. See Tutorial Notes 2. □



Proposition 24. *Let $c(s)$ be a Frenet curve in $\mathbb{R}^3$ parametrized by arc length. Suppose that c lies completely on the unit sphere $\mathbb{S}^2 = \mathbb{S}^2_1(0) \subset \mathbb{R}^3$ centered at origin. Then, the curvature and torsion of c is given by*

$$\kappa = \sqrt{1 + J^2}, \qquad \tau = \frac{J'}{1 + J^2},$$

where $J := \det(c, c', c'')$. The great circles on $\mathbb{S}^2$ are characterized by the condition $J \equiv 0$, and the other circles by $J \equiv$ constant. We call J the geodesic curvature of c in $\mathbb{S}^2$.

Proof. Since c lies on the unit sphere, we have $\langle c, c \rangle \equiv 1$, which implies $\langle c', c \rangle = 0$. Moreover $\langle c', c' \rangle \equiv 1$ since we are using arc length parametrization. Therefore, $\{c, c', c \times c'\}$ forms a positive orthonormal basis (note: this is not the Frenet frame!) and thus we can decompose c'' into components of such a basis. Note that

$$\langle c'', c\rangle = -\langle c', c'\rangle = -1,$$

$$\langle c'', c'\rangle = \frac{1}{2}\langle c', c'\rangle' = 0,$$

$$\langle c'', c \times c'\rangle = \det(c, c', c'') =: J.$$

Therefore, $\kappa = \|c''\| = \sqrt{1+J^2}$. Next, we consider the vector c''', observe that

$$\langle c''', c\rangle = \langle c'', c\rangle' - \langle c'', c'\rangle = 0.$$

Moreover, by the definition of Frenet frame,

$$e_1 = c', \qquad e_2 = \frac{1}{\kappa}c'', \qquad e_3 = e_1 \times e_2 = \frac{1}{\kappa}c' \times c''.$$

Therefore, by the definition of τ,

$$\tau := -\langle e_3', e_2\rangle = -\langle (\frac{1}{\kappa}c' \times c'')', \frac{1}{\kappa}c''\rangle = -\frac{1}{\kappa^2}\langle c' \times c''', c''\rangle = -\frac{1}{\kappa^2}\langle c' \times c''', -c + Jc \times c'\rangle = \frac{1}{\kappa^2}\langle c' \times c''', c\rangle,$$

where the last equality uses the fact that $\langle c''', c\rangle = 0$, which implies that $\langle c' \times c''', c \times c'\rangle = 0$. The formula for τ then follows from the definition of J that

$$J' = \langle c' \times c'', c\rangle' = \langle c' \times c''', c\rangle.$$

The quantity J is in fact an interesting geometric quantity which measures the *curvature of* c *as seen from the sphere* $\mathbb{S}^2$. Note that $c \times c'$ is the unit vector normal to the curve but tangent to the sphere $\mathbb{S}^2$, hence $J = \langle c'', c \times c'\rangle$ is the part of c'' which is tangent to the sphere. From this the rest of the proposition follows easily. (Exercise: can you give a rigorous proof?) $\square$

It is a good exercise for the reader to derive the formulas of κ and τ if the curve lies on a sphere of any radius $R > 0$. Does the center of the sphere matter?

Fundamental theorem of curves in $\mathbb{R}^n$

In this section, we return to the general theory of curves inside $\mathbb{R}^n$. We will derive the Frenet equations for general n and prove the fundamental theorem curves which says that the "curvatures" of the curve uniquely determines a curve in $\mathbb{R}^n$ up to rigid motions.

Proposition 25. *Let $\{e_1, e_2, \cdots, e_n\}$ be the Frenet frame of a Frenet curve c in $\mathbb{R}^n$. Then, they satisfy the Frenet equations below:*

$$\begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ \vdots \\ \vdots \\ e_{n-1} \\ e_n \end{pmatrix}' = \begin{pmatrix} 0 & \kappa_1 & 0 & 0 & \cdots & 0 \\ -\kappa_1 & 0 & \kappa_2 & 0 & \ddots & \vdots \\ 0 & -\kappa_2 & 0 & \ddots & \ddots & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 & \kappa_{n-1} \\ 0 & \cdots & 0 & 0 & -\kappa_{n-1} & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ \vdots \\ \vdots \\ e_{n-1} \\ e_n \end{pmatrix},$$

where $\kappa_i := \langle e_i', e_{i+1}\rangle$, $i = 1, \cdots, n-1$, are the i-th Frenet curvature of the curve c. When $i = n-1$, κ_{n-1} is called the torsion of the curve. Moreover, we have $\kappa_i > 0$ for $i \leq n-2$.

Proof. Recall from the definition of Frenet frame that for $i = 1, 2, \cdots, n-1$, $e_i \in \operatorname{span}\{c', c'', \cdots, c^{(i)}\}$ and thus $e_i' \in \operatorname{span}\{c', c'', \cdots, c^{(i+1)}\} = \operatorname{span}\{e_1, e_2, \cdots, e_{i+1}\}$. This explains the zeros in the upper right corner of the Frenet matrix. The rest of the entries follows from the definition of κ_i and the skew-symmetry: $\langle e_i', e_j\rangle = -\langle e_i, e_j'\rangle$. The positivity of κ_i, $i \leq n-2$, follows from the construction of the Frenet frame e_i using Gram-Schmidt and that the condition $\langle c^{(i)}, e_i\rangle > 0$. □

From above it is easy to see that a Frenet curve in $\mathbb{R}^n$ is contained in a hyperplane if and only if $\kappa_{n-1} \equiv 0$, which is equivalent to saying that e_n is a constant vector (perpendicular to the hyperplane).

As in the case for plane curves, where the curve is uniquely determined by the curvature κ up to rigid motions. We have the same phenomenon in $\mathbb{R}^n$. The lemma below says that the i-th curvatures κ_i is invariant under rigid motions of $\mathbb{R}^n$.

Lemma 26. *Let $c(s)$ be a Frenet curve in $\mathbb{R}^n$. Suppose that $T : \mathbb{R}^n \to \mathbb{R}^n$ is a (orientation preserving) rigid motion of $\mathbb{R}^n$. If we define the curve $\tilde{c} = T \circ c$, then $\tilde{c}$ is a Frenet curve and $\tilde{\kappa}_i = \kappa_i$ for all $i = 1, \cdots, n-1$.*

Proof. Since any rigid motion T of $\mathbb{R}^n$ can be written as a rotation followed by a translations, i.e. $T(x) = Ax + b$ for some *constant* $A \in SO(n)$ and $b \in \mathbb{R}^n$. Since $A^T = A^{-1}$, it is clear that $\tilde{c} := T \circ c$ is still a Frenet curve parametrized by arc length provied c is. Moreover, if $\{e_1, \cdots, e_n\}$ is the Frenet frame of c, then $\{Ae_1, \cdots, Ae_n\}$ is the Frenet frame of $\tilde{c}$ since $\tilde{c}^{(i)} = Ac^{(i)}$ for all i (recall that A is constant!). In addition, the curvatures are the same because

$$(Ae_i)' = Ae_i' = A(-\kappa_{i-1}e_{i-1} + \kappa_i e_{i+1}) = -\kappa_{i-1}(Ae_{i-1}) + \kappa_i(Ae_{i+1}).$$

□

The main theorem for local theory of curves says that we can arbitrarily prescribed the curvatures κ_i of a curve and the curve is determined uniquely up to rigid motions in $\mathbb{R}^n$. We will see later that this is not the same for surfaces in $\mathbb{R}^3$, for example. In the latter case, the "curvatures" have to satisfy some compatibility/integrability conditions called constraint equations in order to guarantee existence. The situation is much simpler for curves because curves do not have any local intrinsic geometry (i.e. any two curves are *locally isometric* to each other).

Theorem 27 (Fundamental Theorem of Curves)**.** *Let $\kappa_1, \kappa_2, \cdots, \kappa_{n-1} : (a,b) \to \mathbb{R}$ be given smooth functions on the interval (a,b) containing 0 and $\kappa_1, \cdots, \kappa_{n-2} > 0$ are positive functions. Let $q \in \mathbb{R}^n$ be a given point and $\{e_1^0, \cdots, e_n^0\}$ be a given n-frame (i.e. positively oriented orthonormal basis). Then, there exists a smooth Frenet curve $c(s) : (a,b) \to \mathbb{R}^n$ parametrized by arc length such that*

(i) $c(0) = q$,

(ii) $\{e_1^0, \cdots, e_n^0\}$ *is the Frenet frame of c at the point q when $s = 0$,*

(iii) $\kappa_1, \kappa_2, \cdots, \kappa_{n-1}$ *are the Frenet curvatures of c.*

Proof. By the existence and uniqueness for linear system of ODEs, the Frenet equations in Proposition **??** is uniquely solvable for all s given initial conditions (ii). Let

$$F = F(s) = \begin{pmatrix} e_1(s) \\ e_2(s) \\ \vdots \\ e_n(s) \end{pmatrix}$$

be an $n \times n$ matrix of functions of s (we think of each e_i as a row vector). Then, the Frenet equation can be written in matrix form as $F' = KF$, where K is the Frenet matrix (depending on s). Note that we get a unique solution to $F' = KF$ with initial condition $F(0) = F_0$ such that $F_0 F_0^T = I$ since $\{e_1^0, \cdots, e_n^0\}$ is a positively oriented orthonormal frame. Therefore, if we define $G(s) = F(s)F(s)^T$, then $G' = F'F^T + F(F')^T = (KF)F^T + F(KF)^T = K(FF^T) + (FF^T)K^T = K(FF^T) - (FF^T)K$ (where we have used the skew-symmetry of the Frenet matrix K). Therefore, G satisfies the following ODE

$$\begin{cases} G' = KG - GK, \\ G(0) = I. \end{cases}$$

Since constant solution is a solution to the initial value problem, by uniqueness, we must have $G(s) \equiv I$ and thus $FF^T = I$ *for all* s. This tells us that the solution $F(s)$ we got by solving the ODEs indeed gives an orthonormal frame *for all* s. Since $\det F \neq 0$ for all s and $\det F(0) = 1$, we have $\det F = 1$ for all s, i.e. $F(s)$ gives a *positively oriented* orthonormal basis. As before, we can integrate out the first row of $F(s)$ to get $c(s)$ with initial condition:

$$\begin{cases} c'(s) = e_1(s), \\ c(0) = q. \end{cases}$$

It is then trivial to verify that $\{e_1(s), \cdots, e_n(s)\}$ is the Frenet frame of the curve $c(s)$ and thus κ_i are the curvatures of c. ☐

Global theory of curves

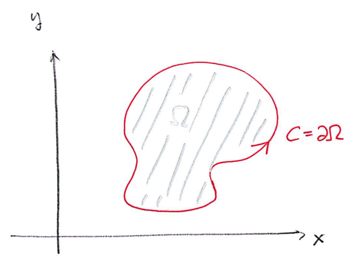
We now turn to the study of the *global* properties of curves. Much of what we have studied so far are *local* properties of curves. A "*local property*" is something which can be determined by information on a small neighborhood of a certain point. For example, the derivative $f'(x_0)$ of a function $f(x)$ at $x = x_0$ is a local quantity since you only need to know the function f near x_0 to be able to find out its derivative at x_0. Another example is that "a curve is regular" is a local property since it can be checked locally (that $c' \neq 0$) at any point. In contrast, "*integration*" is a global property since you need to know the function on the *entire* domain to evaluate the integral. It is very important to understand which properties are local and which are global, and the interplay between local and global quantities.

Every one of us have already learned an important example of such in calculus - *the fundamental theorem of calculus*:

$$\int_a^b f'(x)\, dx = f(b) - f(a),$$

which says that the integral of the derivative of a function depends only on the difference of the values of the function at the end points, *regardless of what the function f is in the interior* (assuming f is C^1, say). The fundamental theorem of calculus has far reaching consequences in analysis and geometry. In particular, one can generalize the situation to the 2D case to obtain the *Green's theorem*:

$$\oint_C P\, dx + Q\, dy = \iint_\Omega \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

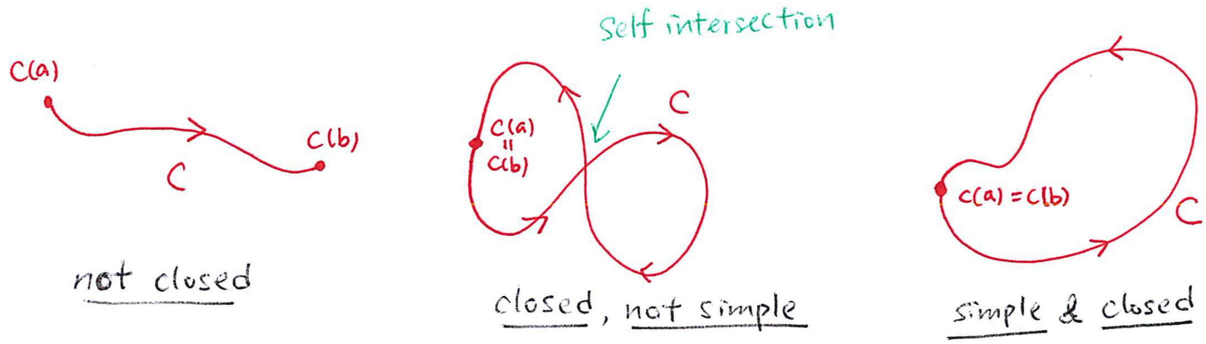


for any C^1 functions P, Q defined on a neighborhood of a smooth (or piecewise C^1) bounded domain $\Omega \subset \mathbb{R}^2$ with $C = \partial\Omega$ (oriented so that the domain Ω always lie on the left of C). Note that the class of plane curves C which is the boundary of some smooth bounded domain $\Omega \subset \mathbb{R}^2$ belongs to an interesting class of objects called *simple closed curves*.

Definition 28. *A (smooth) curve $c : [a, b] \to \mathbb{R}^n$ is <u>closed</u> if $c^{(i)}(a) = c^{(i)}(b)$ for all $i = 0, 1, 2, \cdots$. A closed curve $c : [a, b] \to \mathbb{R}^n$ is <u>simple</u> if c is injective on (a, b).*

Because a (smooth) closed curve c agrees at the end points with all the derivatives, by identifying the end points a and b, one can actually regard it as a smooth map from a circle $\mathbb{S}^1 \approx [a, b]/a \sim b$ to

$\mathbb{R}^n$. A closed curve is then simple if and only if it is bijective as a map from $\mathbb{S}^1$ onto its image (i.e. there is no *self intersections*).



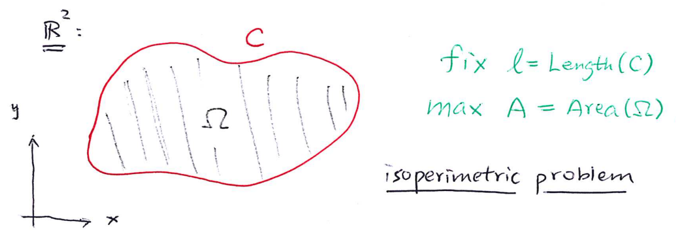
We now focus our attention on simple closed curves in $\mathbb{R}^2$. First, we quote a deep topological theorem about planar closed curves.

Jordan curve theorem: Any (continuous) simple closed curve in $\mathbb{R}^2$ divides the plane into two connected regions: one is bounded (*interior*) and one is unbounded (*exterior*).

Note that a more refined topological result called *Schoenflies theorem* says that the interior is in fact homeomorphic to a disk. These results are highly non-trivial especially in higher dimensions. For example, there exists an embedded surface in $\mathbb{R}^3$ that is homeomorphic to the 2-sphere which bounds an interior region in $\mathbb{R}^3$ not homeomorphic to the 3-ball. A famous example is called the *Alexander's horned sphere.*

However, in the planar case, any simple closed curve C bounds an interior region Ω homeomorphic to the 2-disk. So one can formulate the classical:

Isoperimetric problem: Given a simple closed curve C in $\mathbb{R}^2$ with fixed length ℓ, what is the largest possible area A of the interior region Ω enclosed by the curve C?



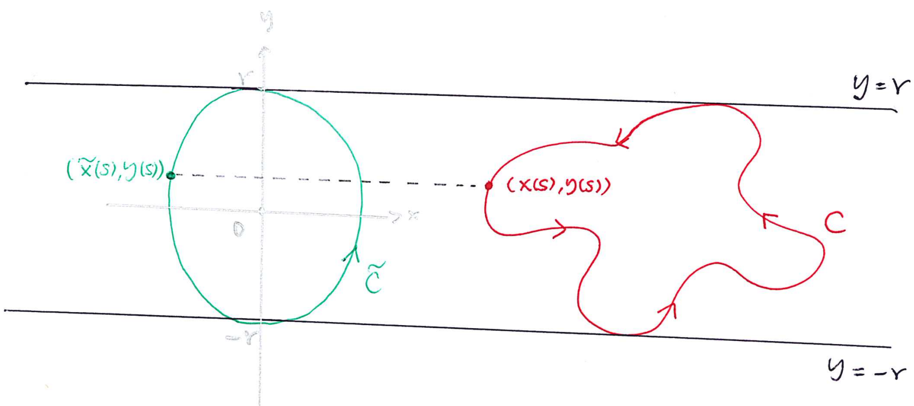
It turns out that the isoperimetric problem in $\mathbb{R}^2$ has a neat answer.

Theorem 29 (The Isoperimetric Inequality)**.** *Let C be a simple closed curve in $\mathbb{R}^2$ with length ℓ, and let A be the area of the interior region Ω bounded by C. Then*

$$A \leq \frac{\ell^2}{4\pi},$$

and equality holds if and only if C is a round circle.

Proof. Let $c(s) = (x(s), y(s)) : [0, \ell] \to \mathbb{R}^2$ be an arc length parametrization of the simple closed curve C, oriented so that the region Ω it bounds stay on the left of C. After a translation and rotation, we can assume that the curve C is contained in the slab $\{|y| \leq r\}$ and it touches the line $\{y = \pm r\}$ at exactly one point. Let $\tilde{C}$ be the circle of radius $r > 0$ centered at origin. See picture below:



Notice that the arc length parametrization of c induces a parametrization (not necessarily arc length parametrization!) of the circle $\tilde{C}$ by a projection perpendicular to the y-axis. Therefore, we have a parametrization of $\tilde{C}$ given by

$$\tilde{c}(s) = (\tilde{x}(s), y(s)), \qquad s \in [0, \ell].$$

Keep in mind that s is not an arc length parameter for the circle $\tilde{C}$. Let A and $\tilde{A}$ be the area of the regions bounded by C and $\tilde{C}$ respectively. Note that $\tilde{A} = \pi r^2$. On the other hand, a simple application of Green's theorem applied to the pair $(P, Q) = (-y, 0), (0, x)$ and $(-y, x)$ implies that if $(x(t), y(t))$, $t \in [a, b]$, is *any* parametrization of a simple closed curve C, then the area of the region Ω bounded by C is given by

$$\text{Area}(\Omega) = -\int_a^b yx'\,dt = \int_a^b xy'\,dt = \frac{1}{2}\int_a^b (xy' - yx')\,dt.$$

Therefore, we have

$$A = -\int_0^\ell y(s)x'(s)\,ds, \qquad \text{and} \qquad \pi r^2 = \tilde{A} = \int_0^\ell \tilde{x}(s)y'(s)\,ds.$$

If we add these two equations (we are omitting the parameter s for simplicity and all the derivatives here are taken with respect to s), we obtain

$$\begin{aligned} A+\pi r^2 &= \int_0^\ell (\tilde{x}y' - yx')\,ds \\ &\leq \int_0^\ell |\langle(\tilde{x},-y),(y',x')\rangle|\,ds \\ &\leq \int_0^\ell \|(\tilde{x},-y)\|\|(y',x')\|\,ds \\ &= \int_0^\ell \|(\tilde{x},-y)\|\,ds = r\ell. \end{aligned} \tag{3}$$

where we have used Cauchy-Schwarz inequality and the facts that $(x(s),y(s))$ is an arc length parametrization for C and that $(\tilde{(x)}(s),-y(s))$ lies on $\tilde{C}$, a circle of radius r centered at origin. By the AM-GM inequality, we obtain

$$\sqrt{\pi r^2 A} \leq \frac{A+\pi r^2}{2} \leq \frac{r\ell}{2}.$$

Rearranging the inequality above yields the desired isoperimetric inequality $A \leq \ell^2/4\pi$.

It remains to study the equality case $A = \ell^2/4\pi$. In this case all the inequalities above are in fact equalities. In particular, the AM-GM inequality tells us that $A = \pi r^2$, which them implies from $A = \ell^2/4\pi$ that $\ell = 2\pi r$. Hence the width of the slab we used to enclosed the curve C is *independent* of the directions of the slab. On the other hand, the equalities in (**??**) implies that for *every* s, we have

$$(\tilde{x}(s),-y(s)) = \lambda(s)(y'(s),x'(s)), \qquad \text{where } \lambda(s) \geq 0.$$

Taking the length of the vectors on both sides, we actually have $\lambda(s) \equiv r > 0$. Therefore, $y(s) \equiv -rx'(s)$. By using instead the slabs $\{|x-a| \leq r\}$ for some $a > 0$ so that the lines $\{x = a \pm r\}$ touches C, we can similarly obtain $x(s) - a \equiv -ry'(s)$ for the same constant $r > 0$. Therefore,

$$(x(s)-a)^2 + y(s)^2 = r^2(x'(s)^2 + y'(s)^2) = r^2,$$

which means that C is the circle of radius $r > 0$. □

Remark 30. *The isoperimetric inequality actually holds for piecewise C^1 curves.*

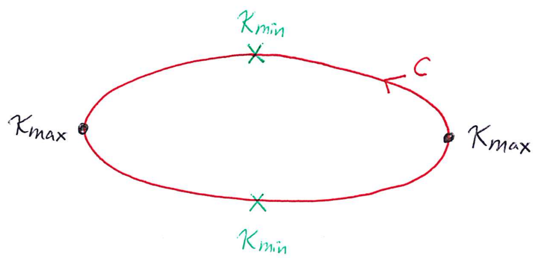
Next, we prove a classical result for planar convex curves regarding the number of "*vertices*". A vertex of a curve c is a point on the curve at which $\kappa' = 0$. A planar simple closed curve is said to be convex if the (closed) region Ω it bounds is a convex subset of $\mathbb{R}^2$, in the sense that the line segment $\overline{pq}$ joining any two points $p,q \in \Omega$ lies entirely inside Ω. One easily check that (Proof?) the following statements are equivalent:

1. The simple closed curve C is convex.
2. If a (infinite) line meets the curve C, the intersection is either a line segment (which could possibly degenerate to a single point) or exactly two points.
3. The curve C always lie on one side of the tangent line at every point on C.

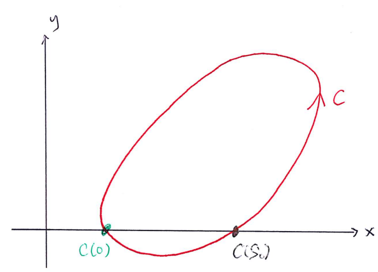
4. The curvature κ of the curve C does not change sign.

Theorem 31 (Four Vertex Theorem)**.** *A simple closed convex planar curve has at least four vertices.*

Note that the number 4 is optimal as one can verify that an ellipse with unequal axes has exactly 4 vertices, two of which are minima and two of which are maxima of κ.



Proof. Without loss of generality, we assume that $\kappa \not\equiv$ constant. Let $c(s) = (x(s), y(s)) : [0, \ell] \to \mathbb{R}^2$ be an arc length parametrization of the curve. Since $\kappa(s)$ is a continuous function on the compact set $[0, \ell]$, it achieves its minimum and maximum so $\kappa' = 0$ there. So there are at least two vertices. Suppose that $\kappa(0) = \min \kappa$ and $\kappa(s_0) = \max \kappa$. Erect a coordinate system such that $c(0)$ and $c(s_0)$ both lie on the x-axis. By the convexity of the curve, the curve meets the x-axis at no other points (unless $\kappa(0) = \kappa(s_0) = 0$ which implies $\kappa(s) \equiv 0$, a contradiction to our assumption). In other words, $y(s)$ changes sign only at $s = 0$ and $s = s_0$.



Next, we argue by contradiction that there are more than two vertices. Suppose not, then there are only two vertices, namely, $c(0)$ and $c(s_0)$. Then, $\kappa'(s)$ changes sign only at $s = 0$ and $s = s_0$, but then the function $\kappa'(s)y(s)$ doesn't change sign at all along the curve. Notice that the Frenet frame is given by

$$e_1 = (x', y'), \qquad e_2 = (-y', x'),$$

and thus the Frenet equation $e_1' = \kappa e_2$ implies that $x'' = -\kappa y'$. Therefore, using this and the closeness of the curve, we get

$$\int_0^\ell \kappa'(s)y(s)\,ds = -\int_0^\ell \kappa(s)y'(s)\,ds = \int_0^\ell x''(s)\,ds = 0.$$

However, we just showed that the integrand $\kappa'(s)y(s)$ has a constant sign, and thus it must vanish identically. This says $\kappa' \equiv 0$, which contradicts our assumption that κ is non-constant. Therefore, there must be a third vertex and κ' must change sign there (why?). But then since the curve is closed, there must be a fourth vertex with a change of sign, which proves the theorem. $\square$

Now, we want to study another global quantity which is the total curvature of a curve. Suppose $c(t) : [a, b] \to \mathbb{R}^2$ is a regular closed plane curve, the total curvature of c is defined to be

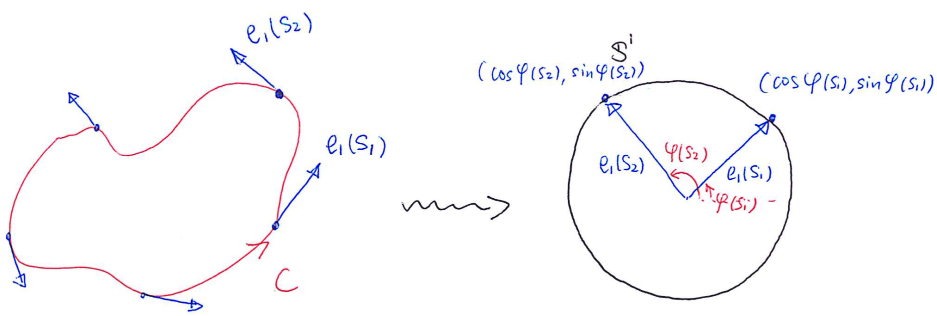
$$TC(c) := \int_a^b \kappa(t) \|c'(t)\| \, dt.$$

If $s \in [0, \ell]$ is an arc length parameter for c, then we have the formula

$$TC(c) = \int_0^\ell \kappa(s) \, ds.$$

Note that $TC(c)$ is independent of the parameterization of c (so it is actually defined on an equivalence class $[c] = \mathcal{C}$). Since changing the orientation of a curve flips the sign of κ, we have $TC(-c) = -TC(c)$ if $-c$ denotes the same curve c with the opposite orientation. We will see that $TC(c)$ is a topological quantity which is invariant under smooth perturbations of the curve c. Suppose $c(s) : [0, \ell] \to \mathbb{R}^2$ is an arc-length parametrized regular plane curve with Frenet frame $\{e_1(s), e_2(s)\}$. Since $e_1(s)$ is a unit vector, we can find some angle $\varphi(s)$ (uniquely determined up to constant integer multiples of 2π) so that

$$e_1(s) = (\cos \varphi(s), \sin \varphi(s)).$$



Therefore, $e_2(s) = (-\sin \varphi(s), \cos \varphi(s))$ and thus

$$\kappa(s) = \langle e_1'(s), e_2(s) \rangle = \varphi'(s).$$

Therefore, geometrically the curvature κ is measuring how fast the unit tangent e_1 is turning (with respect to arc length). Therefore, we have

$$TC(c) = \int_0^\ell \kappa(s) \, ds = \varphi(\ell) - \varphi(0) \in 2\pi\mathbb{Z},$$

since $e_1(0) = e_1(\ell)$ is the same point on $\mathbb{S}^1$. Hence, we define the rotation index of the closed curve c to be the integer

$$U_c := \frac{1}{2\pi} TC(c) = \frac{1}{2\pi} \int_0^\ell \kappa(s)\, ds = \frac{1}{2\pi}(\varphi(\ell) - \varphi(0)).$$

Note that even though the angle $\varphi(s)$ is defined up to an addition constant multiple of 2π, the difference $\varphi(\ell) - \varphi(0)$ is well-defined! Hence, geometrically, the rotation index U_c is measuring how many times have the unit tangent turned a complete circle after traveling along the closed curve c once (note that c may not be simple!). In the case that c is simple, we have the following theorem.

Theorem 32 (Theorem of Turning Tangents)**.** *The rotation index U_c of a simple closed regular plane curve c is ± 1.*

The proof uses strongly the fact that the rotation index U_c is a *homotopy invariant*, i.e. if two closed regular curves c_0 and c_1 can be connected by a smooth family of closed regular curves c_t, $t \in [0, 1]$, then $U_{c_0} = U_{c_1}$. (See Tutorial Notes 3 for the details of the proof.)