THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MMAT 5340 Probability and Stochastic Analysis

Suggested Solution of Homework 2

11.29 Let $x(n) = \begin{bmatrix} P(X_n = 1) & P(X_n = 2) \end{bmatrix}$. Then

$$x(2) = x(0)A^2 = \begin{bmatrix} 0.4 & 0.6 \end{bmatrix} \begin{bmatrix} 0.4 & 0.6 \\ 0.3 & 0.7 \end{bmatrix}^2 = \begin{bmatrix} 0.334 & 0.666 \end{bmatrix}.$$

Hence $P(X_2 = 1) = 0.334$.

11.30 The stationary distribution $p = \begin{bmatrix} p_1 & p_2 \end{bmatrix}$ satisfies

$$p = pA \iff \begin{cases} p_1 = 0.4p_1 + 0.3p_2 \\ p_2 = 0.6p_1 + 0.7p_2 \end{cases} \iff 2p_1 = p_2.$$

Since $p_1 + p_2 = 1$, we have $p = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \end{bmatrix}$.

11.31 Solving

$$0 = |A - \lambda I| = \begin{vmatrix} 0.4 - \lambda & 0.6 \\ 0.3 & 0.7 - \lambda \end{vmatrix} = \lambda^2 - 1.1\lambda + 0.1,$$

the eigenvalues of A are given by

$$\lambda_1 = 1, \quad \lambda_2 = \frac{1}{10}.$$

Hence the rate of convergence is $|\lambda_2|^n = \dfrac{1}{10^n}$.

11.32 Let p_x be the probability that, starting from x, the process hits 2 before 3. Then

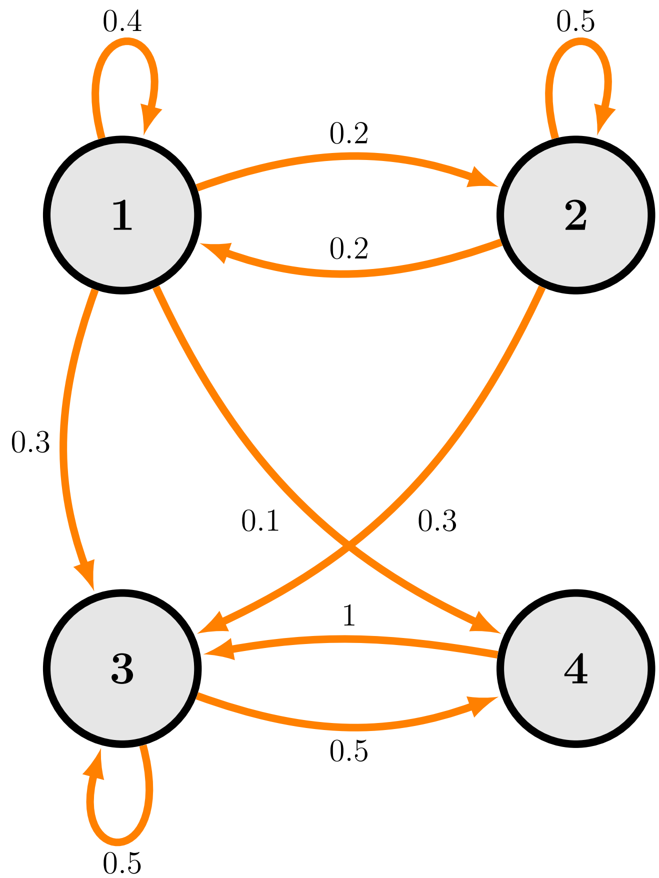
$$p_2 = 1, \quad p_3 = 0,$$

and

$$\begin{aligned} p_1 &= 0.4p_1 + 0.4p_2 + 0.2p_3 \\ &= 0.4p_1 + 0.4 + 0, \end{aligned}$$

so that $p_1 = \dfrac{2}{3}$.

11.36 The graph of the Markov chain is given below:



From the graph, we can see that 1 and 2 are transient states, while 3 and 4 are recurrent states.

11.40 Let $X = Z - 1$. Then

$$P(X = 0) = P(Z = 1) = 0.5, \quad P(X = 1) = P(Z = 2) = 0.5^2 = 0.25,$$
$$P(X \geq 2) = 1 - P(X = 0) - P(X = 2) = 0.25$$

Hence, the transition matrix is

$$P = \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0.5 & 0.25 & 0.25 \\ 0 & 0.5 & 0.5 \end{bmatrix}$$

We can find the stationary distribution $p = \begin{bmatrix} p_0 & p_1 & p_2 \end{bmatrix}$ by solving $pP = p$: By performing column operations,

$$P - I = \begin{bmatrix} -0.5 & 0.5 & 0 \\ 0.5 & -0.75 & 0.25 \\ 0 & 0.5 & -0.5 \end{bmatrix} \sim \begin{bmatrix} -0.5 & 0 & 0 \\ 0.5 & -0.25 & 0.25 \\ 0 & 0.5 & -0.5 \end{bmatrix}$$
$$\sim \begin{bmatrix} -0.5 & 0 & 0 \\ 0.5 & -0.25 & 0 \\ 0 & 0.5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -2 & 0 \end{bmatrix}.$$

That is $p_0 = p_1 = 2p_2$. Hence $p = \begin{bmatrix}0.4 & 0.4 & 0.2\end{bmatrix}$.

The long-term average premium is

$$\begin{aligned} p_0 r_0 + p_1 r_1 + p_2 r_2 &= (0.4)(0.5 \cdot 0 + 1) + (0.4)(0.5 \cdot 1 + 1) + (0.2)(0.5 \cdot 2 + 1) \\ &= 1.4 \text{ thousands of dollars.} \end{aligned}$$

12.12 Let n and m be the quantity of up and down steps. Then

$$\begin{cases} n + m = 13 - 2 = 11 \\ n - m = 3 - 2 = 1. \end{cases}$$

Hence $n = 6, m = 5$. Thus

$$P(S_{13} = 3 | S_2 = 2) = \binom{11}{6} (0.7)^6 (0.3)^5 \approx 0.1321.$$

12.13 By the reflection principle, every path from $(2, 2)$ to $(13, 3)$ which hits $y = 1$ corresponds to a path from $(2, 2)$ to $(13, -1)$. Solving

$$\begin{cases} n + m = 13 - 2 = 11 \\ n - m = -1 - 2 = -3, \end{cases}$$

we have $n = 4, m = 7$. The number of such path is $\binom{11}{4}$. Thus

$$P(S_{13} = 3,\ S_n > 1,\ n = 2, 3, \dots, 13 | P_2 = 2) = \left[\binom{11}{6} - \binom{11}{4}\right] (0.7)^6 (0.3)^5 \approx 0.0377.$$

12.14 Solving

$$\begin{cases} n + m = 10 - 0 = 10 \\ n - m = 2 - 0 = 2, \end{cases} \implies \begin{cases} n = 6 \\ m = 4. \end{cases}$$

So there are $\binom{10}{6}$ paths from $(0, 0)$ to $(10, 2)$. By the reflection principle, every path from $(0, 0)$ to $(10, 2)$ which hits $y = -2$ corresponds to a path from $(0, 0)$ to $(10, -2 - (2 - (-2))) = (10, -6)$. Solving

$$\begin{cases} n + m = 10 - 0 = 10 \\ n - m = -6 - 0 = -6, \end{cases} \implies \begin{cases} n = 2 \\ m = 8. \end{cases}$$

So there are $\binom{10}{2}$ paths from $(0, 0)$ to $(10, -6)$. Thus,

$$P(S_{10} = 2,\ S_n > -2,\ n = 0, \dots, 10) = \left[\binom{10}{6} - \binom{10}{2}\right] (0.5)^{10} \approx 0.1611.$$

12.27 The risk-neutral probability p_0, q_0 can be found by $\mathbf{E}_0 P_1 = P_0$:

$$\begin{cases} p_0 \cdot 2 + q_0 \cdot 0.3 = 1 \\ p_0 + q_0 = 1 \end{cases} \implies p_0 = \frac{7}{17},\ q_0 = \frac{10}{17}.$$

Hence, the fair price is given by

$$\begin{aligned} v &= \mathbf{E}_0 (P_3 - K)_+ \\ &= p_0^3 (2^3 - 1) + 3p_0^2 q_0 (2^2 \cdot 0.3 - 1) + 3p_0 q_0^2 (0) + q_0^3 (0) \\ &\approx 0.5485. \end{aligned}$$

13.20 Since X_n is the number of Heads during the first n tosses, we have

$$\mathbf{E}(X_{n+1} \mid X_n) = \frac{1}{2}(X_n + 1) + \frac{1}{2}X_n = X_n + \frac{1}{2}.$$

If $Y_n := 3X_n - cn$ is a martingale, then

$$\begin{aligned} Y_n = \mathbf{E}(Y_{n+1} \mid Y_0, \dots, Y_n) &= \mathbf{E}(Y_{n+1} \mid X_0, \dots, X_n) \\ &= \mathbf{E}(3X_{n+1} - c(n+1) \mid X_n) \\ &= 3\mathbf{E}(X_{n+1} \mid X_n) - c(n+1) \\ &= 3X_n + \frac{3}{2} - c(n+1) \\ &= Y_n + \frac{3}{2} - c. \end{aligned}$$

Hence $c = \dfrac{3}{2}$.

13.30 Need to find c such that $M_n = e^{S_n - cn}$ is a martingale:

$$\mathbf{E}e^{-1+X_1-c} = e^{-1} \implies e^{4/2} = F_{N(0,4)}(1) = e^c \implies c = 2.$$

For $x > 0$, $f(x) = x^3$ is convex, since $f''(x) = 6x > 0$. Note that $M_n > 0$. By Doob's martingale inequality, for $\lambda > 0$, we have,

$$\mathbf{P}\left(\max_{0 \le n \le 100} M_n \ge \lambda\right) \le \frac{\mathbf{E}M_{100}^3}{\lambda^3}.$$

Now

$$M_{100}^3 = e^{-3}e^{3S_{100}}e^{-300c} = e^{-3}e^{-300c}e^{3X_1} \cdots e^{3X_{100}},$$

so that

$$\mathbf{E}M_{100}^3 = e^{-3}e^{-300c}(\mathbf{E}e^{3X_1})^{100} = e^{-3}e^{-300(2)}\left(e^{\frac{(3)^2 4}{2}}\right)^{100} = e^{1197}.$$

Hence

$$\mathbf{P}\left(\max_{0 \le n \le 100} M_n \ge \lambda\right) \le \frac{e^{1197}}{\lambda^3}.$$

14.12 Note that $\tau_4 - \tau_3$ and $\tau_3 - \tau_2$ are i.i.d. $\text{Exp}(\lambda)$ random variables, with $\lambda = 3$. Hence

$$\tau_4 - \tau_2 = (\tau_4 - \tau_3) + (\tau_3 - \tau_2) \sim \Gamma(2, \lambda) = \Gamma(2, 3).$$

Therefore,

$$\mathbf{E}(\tau_4 - \tau_2) = 2\lambda^{-1} = \frac{2}{3},$$

$$\text{Var}(\tau_4 - \tau_2) = 2\lambda^{-2} = \frac{2}{9}.$$

14.13 Since $N(t) - N(s)$ is independent of $N(u), u \le s$, and $N(t) - N(s) \sim \text{Poi}(\lambda(t-s))$, we have

$$\begin{aligned}\mathbf{P}(N(5/2) = 3 \mid N(1) = 1) &= \mathbf{P}(N(5/2) - N(1) = 2 \mid N(1) = 1) \\ &= \mathbf{P}(N(5/2) - N(1) = 2) \\ &= \frac{(\lambda(3/2))^2}{2!} e^{-\lambda(3/2)} \\ &= \frac{81}{8} e^{-9/2}.\end{aligned}$$

14.18 Note that $\mathbf{E}N(t) = \text{Var}N(t) = \lambda t = t$. By (35) and (36),

$$\mathbf{E}X(t) = \mathbf{E}N(t) \cdot \mathbf{E}Z_k = t \cdot 2 = 2t,$$

$$\text{Var}X(t) = \mathbf{E}N(t) \cdot \text{Var}Z_k + \text{Var}N(t) \cdot (\mathbf{E}Z_k)^2 = t \cdot 3 + t \cdot 2^2 = 7t.$$

15.24 The stationary distribution $\rho = \begin{bmatrix}\rho_1 & \rho_2 & \rho_3\end{bmatrix}$ satisfies $\rho A = 0$. Since $\rho_1 + \rho_2 + \rho_3 = 1$, we have

$$\rho = \frac{1}{1+5+2}\begin{bmatrix}1 & 5 & 2\end{bmatrix} = \begin{bmatrix}\frac{1}{8} & \frac{5}{8} & \frac{1}{4}\end{bmatrix}.$$

15.25 Since $0 > \lambda_2 > \lambda_3$, the rate of convergence is $e^{\lambda_2 t} = e^{-2t}$.

15.26 The transition matrix for the corresponding discrete-time Markov chain is

$$P = \begin{bmatrix}0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 \\ \frac{1}{3} & \frac{2}{3} & 0\end{bmatrix}.$$

15.27 Let λ_i', $1 \le i \le 3$, be the intensity of exit from state i, that is, the negative of the diagonal entries of A. Then the stationary distribution π for the corresponding discrete-time Markov chain is given by

$$\begin{aligned}\pi &= \frac{1}{\lambda_1'\rho_1 + \lambda_2'\rho_2 + \lambda_3'\rho_3}\begin{bmatrix}\lambda_1'\rho_1 & \lambda_2'\rho_2 & \lambda_3'\rho_3\end{bmatrix} \\ &= \frac{8}{13}\begin{bmatrix}\frac{1}{4} & \frac{5}{8} & \frac{3}{4}\end{bmatrix} \\ &= \begin{bmatrix}\frac{2}{13} & \frac{5}{13} & \frac{6}{13}\end{bmatrix}.\end{aligned}$$