**MATH 4030 Differential Geometry**
**Homework 9**
*due 15/11/2015 (Tue) at 5PM*

# Problems

(to be handed in)

1. Suppose $X(u,v) : U \to S \subset \mathbb{R}^3$ is an *orthogonal* parametrization of a surface $S$ such that the first fundamental form is diagonal:
$$(g_{ij}) = \begin{pmatrix} E & 0 \\ 0 & G \end{pmatrix}.$$
Show that the Gauss curvature is given by
$$K = -\frac{1}{2\sqrt{EG}} \left[ \left( \frac{E_v}{\sqrt{EG}} \right)_v + \left( \frac{G_u}{\sqrt{EG}} \right)_u \right].$$
If, in addition, that $X$ is an *isothermal* parametrization, i.e. $E = G = \lambda(u,v)$, then show that
$$K = -\frac{1}{2\lambda} \Delta(\log \lambda)$$
where $\Delta := \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}$ is the standard Euclidean Laplace operator.

2. Prove that the surfaces parametrized by $(u,v) \in (0,+\infty) \times (0,2\pi)$,
$$X(u,v) = (u\cos v, u\sin v, \log u)$$
$$\tilde{X}(u,v) = (u\cos v, u\sin v, v)$$
have the same Gauss curvature at the points $X(u,v)$ and $\tilde{X}(u,v)$. However, show that the map $\tilde{X} \circ X^{-1}$ is not an isometry.

3. Compute all the Christoffel symbols for the subset of the plane parametrized in
   (a) rectangular coordinates: $X(x,y) = (x,y,0)$, where $(x,y) \in \mathbb{R}^2$,
   (b) polar coordinates: $\tilde{X}(r,\theta) = (r\cos\theta, r\sin\theta, 0)$, where $(r,\theta) \in (0,+\infty) \times (0,2\pi)$.

4. Show that no neighborhood of a point on the unit sphere $\mathbb{S}^2$ is isometric to a subset of the plane.

# Suggested Exercises

(no need to hand in)

1. Compute all the Christoffel symbols of a surface of revolution parametrized by
$$X(u,v) = (f(u)\cos v, f(v)\cos u, g(v)), \quad (u,v) \in (0,2\pi) \times \mathbb{R}$$
where $f, g : \mathbb{R} \to \mathbb{R}$ are smooth functions with $f > 0$ everywhere.

2. Explain why the saddle surface $\{z = x^2 - y^2\}$ is not locally isometric to any round sphere or cylinder.