MATH 4030 Differential Geometry
Homework 8
due 8/11/2015 (Tue) at 5PM

Problems

(to be handed in)

Unless otherwise stated, we use U, O to denote connected open subsets of $\mathbb{R}^n$. The symbols $S, \tilde{S}, S_1, S_2, S_3$ always denote surfaces in $\mathbb{R}^3$. Recall that a *critical point* of a smooth function $f : S \to \mathbb{R}$ is a point $p \in S$ such that $df_p = 0$.

1. Let $f : S_1 \to S_2$ be an isometry between two compact surfaces S_1, S_2 in $\mathbb{R}^3$. Show that S_1 and S_2 have the same area. *(You can assume that S_2 is covered by a single parametrization except a set of measure zero.)*

2. Let $S_1 = \{z = 0\}$ be the xy-plane and $S_2 = \{x^2 + y^2 = 1\}$ be the right unit cylinder. Show that the map $f : S_1 \to S_2$ defined by $f(x, y, 0) = (\cos x, \sin x, y)$ is a local isometry.

3. Suppose $X : U \to V \subset S$ and $\tilde{X} : U \to \tilde{V} \subset \tilde{S}$ are parametrizations of two surfaces $S, \tilde{S}$ in $\mathbb{R}^3$ such that their first fundamental forms are the same, i.e. for $i, j = 1, 2$,

$$g_{ij} = \langle \frac{\partial X}{\partial u_i}, \frac{\partial X}{\partial u_j} \rangle = \langle \frac{\partial \tilde{X}}{\partial u_i}, \frac{\partial \tilde{X}}{\partial u_j} \rangle = \tilde{g}_{ij} \qquad \text{on } U.$$

   Show that V is isometric to $\tilde{V}$.

4. Find a local isometry $f : S_1 \to S_2$ from the upper half plane $S_1 = \{z = 0, y > 0\}$ to the cone $S_2 := \{x^2 + y^2 = z^2, z > 0\}$. Calculate the mean and Gauss curvatures of S_2.

Suggested Exercises

(no need to hand in)

1. Given a surface $S \subset \mathbb{R}^3$, prove that the set of isometries $f : S \to S$ form a group under composition. This is called the *isometry group of* S. What is the isometry group of the unit sphere $\mathbb{S}^2 = \{x^2 + y^2 + z^2 = 1\}$?

2. Find a local isometry between the helicoid and the catenoid. Are they globally isometric?

3. Let $\alpha(t) : (-\epsilon, \epsilon) \to S$ be a curve on a surface $S \subset \mathbb{R}^3$. Suppose $X(t), Y(t)$ are two tangential vector fields defined along the curve α. Prove that

$$\frac{d}{dt}\langle X(t), Y(t) \rangle = \langle \nabla_{\alpha'(t)} X(t), Y(t) \rangle + \langle X(t), \nabla_{\alpha'(t)} Y(t) \rangle.$$

   Using this result, prove that the angle between two parallel vector fields X, Y along a curve is always constant.