**MATH 4030 Differential Geometry**
**Homework 7**
*due 1/11/2015 (Tue) at 5PM*

# Problems

(to be handed in)

Unless otherwise stated, we use $U, O$ to denote connected open subsets of $\mathbb{R}^n$. The symbols $S, S_1, S_2, S_3$ always denote surfaces in $\mathbb{R}^3$. Recall that a *critical point* of a smooth function $f : S \to \mathbb{R}$ is a point $p \in S$ such that $df_p = 0$.

1. Let $S \subset \mathbb{R}^3$ be a surface. Fix $p_0 \in \mathbb{R}^3$ and consider the smooth function $f : S \to \mathbb{R}$ defined by $f(p) = |p - p_0|^2$.
   (a) Show that $p \in S$ is a critical point of $f$ if and only if $p_0$ lies on the *normal line* of $S$ at $p$, i.e.
   $$p_0 = p + \lambda N(p)$$
   for some $\lambda \in \mathbb{R}$, here $N(p)$ is *any* unit normal to $S$ at $p$.
   (b) Calculate the hessian of $f$ at a critical point $p \in S$ and show that
   $$d^2 f_p(v) = 2\big(|v|^2 - \lambda A(v, v)\big),$$
   where $\lambda$ is the constant as in (a) and $A$ is the second fundamental form of $S$ at $p$ (relative to the normal $N$ as in (a)).

2. Show that there is no compact surface $S \subset \mathbb{R}^3$ with $K < 0$ everywhere.

3. (a) Let $p$ be a point on a surface $S \subset \mathbb{R}^3$. Prove that $K(p) > 0$ if and only if there exists a point $p_0 \in \mathbb{R}^3$ such that $p$ is a local maximum of the function $f(x) = |x - p_0|^2$.
   (b) Show that there is no compact surface $S \subset \mathbb{R}^3$ with $K \leq 0$ everywhere.

4. Compute the mean curvature $H$ and Gauss curvature $K$ of the *catenoid* given by the parametrization
   $$X(u, v) = (\cosh v \cos u, \cosh v \sin u, v), \qquad (u, v) \in (0, 2\pi) \times \mathbb{R}.$$

5. Show that a graphical surface $S = \{z = f(x, y)\}$ is *minimal* (i.e. $H \equiv 0$) if and only if $f$ satisfies the *minimal surface equation*:
   $$(1 + f_x^2) f_{yy} - 2 f_x f_y f_{xy} + (1 + f_y^2) f_{xx} = 0.$$

## Suggested Exercises

(no need to hand in)

1. Compute the mean curvature $H$ and Gauss curvature $K$ of the torus of revolution given by the parametrization

$$X(u,v) = ((a+b\cos u)\cos v, (a+b\cos u)\sin v, b\sin u), \qquad (u,v) \in (0,2\pi)\times(0,2\pi),$$

   where $a > b > 0$ are constants.

2. Describe the region of $\mathbb{S}^2$ covered by the image of the Gauss map of the following surfaces:
   (a) Paraboloid of revolution $z = x^2 + y^2$.
   (b) Hyperboloid of revolution $x^2 + y^2 - z^2 = 1$.
   (c) Catenoid $x^2 + y^2 = \cosh^2 z$.

3. Determine all the umbilic points on the ellipsoid (where $a > b > c > 0$ are distinct positive real numbers):

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

   At which point(s) does the Gauss curvature $K$ attains its maximum? What about for the mean curvature $H$ (assuming the orientation is chosen such that $H > 0$)?