MATH 4030 Differential Geometry
Homework 6
due 25/10/2015 (Tue) at 5PM

Problems

(to be handed in)

Unless otherwise stated, we use U, O to denote connected open subsets of $\mathbb{R}^n$. The symbols S, S_1, S_2, S_3 always denote surfaces in $\mathbb{R}^3$. Recall that a *critical point* of a smooth function $f : S \to \mathbb{R}$ is a point $p \in S$ such that $df_p = 0$.

1. Find the area of the torus of revolution S defined by

$$S = \{(x, y, z) \in \mathbb{R}^3 : (\sqrt{x^2 + y^2} - a)^2 + z^2 = r^2\},$$

   where $a > r > 0$ are given positive constants.

2. Calculate the mean curvature H and Gauss curvature K of the following surfaces:

$$S_1 = \{(x, y, z) \in \mathbb{R}^3 : z = x^2 + y^2\},$$

$$S_2 = \{(x, y, z) \in \mathbb{R}^3 : z = x^2 - y^2\},$$

   with respect to the "upward" (toward positive z-axis) pointing unit normal N. Express the second fundamental form A of each surface at $p = (0, 0, 0)$ as a diagonal matrix. What are the principal curvatures and principal directions? Sketch the surfaces near $(0, 0, 0)$.

3. (*Invariance under rigid motions*) Let $S \subset \mathbb{R}^3$ be an orientable surface and $\phi : \mathbb{R}^3 \to \mathbb{R}^3$ be a rigid motion of $\mathbb{R}^3$, i.e. $\phi(p) = Ap + b$ for some $A \in O(3)$ and $b \in \mathbb{R}^3$. Let $S' = \phi(S)$ be the image surface of S under ϕ.

   (a) If $N : S \to \mathbb{S}^2$ is a Gauss map for S, prove that $N' = A(N \circ \phi^{-1}) : S' \to \mathbb{S}^2$ is a Gauss map for S'.

   (b) Let A and A' be the second fundamental for S and S' respectively (with respect to N and N' in (a)). Show that for any $p \in S$ and $v, w \in T_pS$,

$$A'_{\phi(p)}(d\phi_p(v), d\phi_p(w)) = A_p(v, w).$$

   (c) Find the relation between the mean curvature and Gauss curvature of S and S'.

Suggested Exercises

(no need to hand in)

1. Show that an orientable connected surface S with $H \equiv 0 \equiv K$ must be contained in a plane.

2. Show that an *ellipsoid*

$$S = \left\{ (x, y, z) \in \mathbb{R}^3 : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \right\},$$

   where $a, b, c > 0$ are constants, has positive Gauss curvature at every point.

3. Suppose that a surface S and a plane P are tangent along the trace of a regular curve $\alpha : I \to \mathbb{R}^3$ with $\alpha(I) \subset P$. Show that the Gauss curvature of S vanishes at every point on this curve.

4. If a surface S contains a straight line $\ell \subset S$, show that the Gauss curvature K of S satisfies $K \leq 0$ at every point on ℓ.