MATH 4030 Differential Geometry
Homework 5
due 11/10/2015 (Tue) at 5PM

Problems

(to be handed in)

Unless otherwise stated, we use U, O to denote connected open subsets of $\mathbb{R}^n$. The symbols S, S_1, S_2, S_3 always denote surfaces in $\mathbb{R}^3$. Recall that a *critical point* of a smooth function $f : S \to \mathbb{R}$ is a point $p \in S$ such that $df_p = 0$.

1. Let $f, g : \mathbb{R} \to \mathbb{R}$ be smooth functions such that $f(u) > 0$ and $g'(u) > 0$ for all $u \in \mathbb{R}$. Define a smooth map $X : \mathbb{R} \times (-\pi, \pi) \to \mathbb{R}^3$ by
$$X(u, v) := (f(u) \cos v, f(u) \sin v, g(u)).$$
Show that the image $S := X(\mathbb{R} \times (-\pi, \pi))$ is a surface, and that all the normal lines of S pass through the z-axis. What does the surface S look like?

2. Let $S \subset \mathbb{R}^3$ be a surface.
   (a) Fix a point $p_0 \in \mathbb{R}^3$ such that $p_0 \notin S$, show that the *distance function from* p_0
   $$f(p) := |p - p_0|$$
   defines a smooth function $f : S \to \mathbb{R}$. Moreover, prove that $p \in S$ is a critical point of f if and only if the line joining p to p_0 is normal to S at p.
   (b) Fix a unit vector $v \in \mathbb{R}^3$, show that the *height function along* v
   $$h(p) := \langle p, v \rangle$$
   defines a smooth function $h : S \to \mathbb{R}$. Prove that $p \in S$ is a critical point of h if and only if v is normal to S at p.

3. Prove the *Chain Rule*: if $f : S_1 \to S_2$ and $g : S_2 \to S_3$ are smooth maps between surfaces, then for any $p \in S_1$,
$$d(g \circ f)_p = dg_{f(p)} \circ df_p.$$

4. Let $a \in \mathbb{R}$ be a regular value of a smooth function $F : O \subset \mathbb{R}^3 \to \mathbb{R}$. Prove that the surface $S = F^{-1}(a)$ is orientable.

Suggested Exercises

(no need to hand in)

1. Let S be a compact connected surface, and $a \in \mathbb{R}^3$ be a unit vector.
   (a) Prove that there exists a point on S whose normal line is parallel to a.
   (b) Suppose all the normal lines of S are parallel to a. Show that S is contained in some plane orthogonal to a.
2. Let $f : S_1 \to S_2$ be a diffeomorphism between surfaces.
   (a) Show that orientability is preserved by diffeomorphisms, i.e. show that S_1 is orientable if and only if S_2 is orientable.
   (b) (*Orientation-reversing diffeomorphism*) Show that the antipodal map $f : \mathbb{S}^2 \to \mathbb{S}^2$ defined by $f(x) = -x$ is an orientation-reversing diffeomorphism on the unit sphere $\mathbb{S}^2$ in $\mathbb{R}^3$.
3. Let $f : S_1 \to S_2$ be a smooth map between surfaces where S_2 is orientable. If f is a local diffeomorphism at every $p \in S_1$, prove that S_1 is also orientable.
4. (*Change of coordinates for tangent vectors*) Let $p \in S$ be a point on the surface S. Suppose there are two parametrizations

$$X(u,v) : U \subset \mathbb{R}^2 \to V \subset S,$$

$$\overline{X}(\overline{u},\overline{v}) : \overline{U} \subset \mathbb{R}^2 \to \overline{V} \subset S$$

such that $p \in V \cap \overline{V}$. Let $\psi = \overline{X}^{-1} \circ X : X^{-1}(V \cap \overline{V}) \to \overline{X}(V \cap \overline{V})$ be the transition map which can be written in (u,v) and $(\overline{u},\overline{v})$ coordinates as

$$\psi(u,v) = (\overline{u}(u,v), \overline{v}(u,v)).$$

If $v \in TpS$ is a tangent vector which can be expressed in local coordinates as

$$a_1 \frac{\partial X}{\partial u} + a_2 \frac{\partial X}{\partial v} = v = b_1 \frac{\partial \overline{X}}{\partial \overline{u}} + b_2 \frac{\partial \overline{X}}{\partial \overline{v}},$$

where the partial derivatives are evaluated at the point $X^{-1}(p)$ and $\overline{X}^{-1}(p)$ respectively, then

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} \frac{\partial \overline{u}}{\partial u} & \frac{\partial \overline{u}}{\partial v} \\ \frac{\partial \overline{v}}{\partial u} & \frac{\partial \overline{v}}{\partial v} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix},$$

where the partial derivatives are evaluated at the point $(u_0, v_0) = X^{-1}(p)$. In other words, the tangent vector transforms by multiplying the *Jacobian matrix* of the transition map ψ.