MATH 4030 Differential Geometry
Homework 4
due 4/10/2015 (Tue) at 5PM

Problems

(to be handed in)

1. Show that the two-sheeted cone $S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = z^2\}$ is not a surface.
2. Let $F(x, y, z) = z^2$. Prove that 0 is not a regular value of F but $F^{-1}(0)$ is a surface.
3. Show that $S = \{(x, y, z) \in \mathbb{R}^3 : z = x^2 - y^2\}$ is a surface and that the following are parametrizations for S:
$$X_1(u, v) = (u + v, u - v, 4uv), \quad (u, v) \in \mathbb{R}^2$$
$$X_2(u, v) = (u \cosh v, u \sinh v, u^2), \quad (u, v) \in \mathbb{R}^2, u \neq 0.$$
4. Determine the tangent planes of the surface $S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 - z^2 = 1\}$ at the points $(x, y, 0) \in S$ and show that they are all parallel to the z-axis.
5. If all normal lines to a connected surface S passes through a fixed point $p_0 \in \mathbb{R}^3$, show that S is contained in a sphere.

Suggested Exercises

(no need to hand in)

1. Let $S = \{p \in \mathbb{R}^3 : |p|^2 - \langle p, a\rangle^2 = r^2\}$ with $|a| = 1$ and $r > 0$ be a right cylinder of radius r whose axis is the line passing through the origin with direction a. Prove that
$$T_pS = \{v \in \mathbb{R}^3 : \langle p, v\rangle - \langle p, a\rangle\langle a, v\rangle = 0\}.$$
Conclude that all the normal lines of S cut the axis orthogonally. Prove the converse as well: i.e. if S is a connected surface whose normal lines all intersect a fixed straight line $\ell \subset \mathbb{R}^3$ orthogonally, then S is a subset of a right cylinder with axis ℓ.
2. Construct an explicit diffeomorphism between the ellipsoid $S_1 = \{(x, y, z) \in \mathbb{R}^3 : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1\}$ and the sphere $S_2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$.
3. Show that each of the subset $(a, b, c, \neq 0)$ below are surfaces
$$S_1 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = ax\},$$
$$S_2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = by\},$$
$$S_3 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = cz\}.$$
Prove that they all intersect orthogonally.

4. Let $S \subset \mathbb{R}^3$ be a surface. Suppose $P \subset \mathbb{R}^3$ is a plane such that S lies on one side of P, show that $T_qP = T_qS$ at all $q \in P \cap S$.

5. Let S be a compact surface, show that there exists a straight line ℓ cutting S orthogonally at at least two points.