MATH 4030 Differential Geometry
Homework 3
due 27/9/2015 (Tue) at 5PM

Problems

(to be handed in)

Unless otherwise stated, we use I, J to denote connected open intervals in $\mathbb{R}$. We use k and τ to denote the curvature and torsion of a regular curve $\alpha : I \to \mathbb{R}^3$.

1. Let $\alpha : I \to \mathbb{R}^3$ be a space curve p.b.a.l. with $k(s) > 0$ for all $s \in I$. Show that
   (a) α lies on a plane if and only if $\tau(s) \equiv 0$ for all $s \in I$.
   (b) α is an arc of a circular helix or circle if and only if both k and τ are constant.
2. Let $\alpha : I \to \mathbb{R}^3$ be a space curve p.b.a.l. and $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$ be a rigid motion of $\mathbb{R}^3$. If we let $\beta = \varphi \circ \alpha : I \to \mathbb{R}^3$, which is also a space curve p.b.a.l., show that $k_\beta(s) = k_\alpha(s)$ and
$$\tau_\beta(s) = \begin{cases} \tau_\alpha(s) & \text{if } \varphi \text{ is orientation-preserving,} \\ -\tau_\alpha(s) & \text{if } \varphi \text{ is orientation-reversing.} \end{cases}$$
3. Let $\alpha : I \to \mathbb{R}^3$ be a regular curve (not necessarily p.b.a.l.), show that the curvature and torsion of α at $t \in I$ is given respectively by
$$k(t) = \frac{|\alpha'(t) \times \alpha''(t)|}{|\alpha'(t)|^3} \quad \text{and} \quad \tau(t) = -\frac{(\alpha'(t) \times \alpha''(t)) \cdot \alpha'''(t)}{|\alpha'(t) \times \alpha''(t)|^2}.$$
4. Let $\alpha : I \to \mathbb{R}^3$ be a space curve p.b.a.l. with $k(s) > 0$, $k'(s) \neq 0$ and $\tau(s) \neq 0$ for all $s \in I$. Prove that the trace of α is contained in a sphere of radius $r > 0$ if and only if
$$\frac{1}{k(s)^2} + \frac{k'(s)^2}{k(s)^4 \tau(s)^2} \equiv r^2.$$

Suggested Exercises

(no need to hand in)

1. Let $\alpha : I \to \mathbb{R}^2$ be a regular plane curve described in polar coordinates by $r = r(\theta)$, i.e.
$$\alpha(\theta) = (r(\theta)\cos\theta, r(\theta)\sin\theta), \quad \theta \in I$$
   (a) Show that for any $[a, b] \subset I$, $L_a^b(\alpha) = \int_a^b \sqrt{r(\theta)^2 + r'(\theta)^2}\, d\theta$.

(b) Show that the curvature at $\theta \in I$ is given by

$$k(\theta) = \frac{2r'(\theta)^2 - r(\theta)r''(\theta) + r(\theta)^2}{[r'(\theta)^2 + r(\theta)^2]^{3/2}}.$$

2. Let $\alpha : I \to \mathbb{R}^2$ be a plane curve p.b.a.l.. Suppose that

$$|\alpha(s_0)| = \max_{s \in I} |\alpha(s)|$$

for some $s_0 \in I$. Prove that $|k(s_0)| \geq 1/|\alpha(s_0)|$. Can one say anything if "maximum" is replaced by "minimum"?

3. (*Lancret's theorem*) A space curve $\alpha : I \to \mathbb{R}^3$ p.b.a.l. with $k(s) > 0$ for all $s \in I$ is said to be a *helix* if all of its tangent lines make a constant angle with a given direction, i.e. there exists some unit vector $v \in \mathbb{R}^3$ such that $\alpha'(s) \cdot v \equiv$ constant. Prove that α is a helix if and only if there exists a constant $c \in \mathbb{R}$ such that $\tau(s) = ck(s)$ for all $s \in I$.

4. Let $\alpha : I \to \mathbb{R}^3$ be a space curve p.b.a.l. with $k(s) > 0$ and $\tau(s) \neq 0$ for all $s \in I$. We call α a *Bertrand curve* if there exists another space curve $\beta : I \to \mathbb{R}^3$ p.b.a.l. such that the normal lines of α and β coincide for each $s \in I$. In this case, we can write

$$\beta(s) = \alpha(s) + r(s)N_\alpha(s).$$

Show that $r(s) \equiv$ constant. Moreover, prove that α is a *Bertrand curve* if and only if

$$Ak_\alpha(s) + B\tau_\alpha(s) \equiv 1$$

for some nonzero constants $A, B \in \mathbb{R}$.