MATH 4030 Differential Geometry
Homework 10
due 22/11/2015 (Tue) at 5PM

Problems

(to be handed in)

1. Does there exist a parametrization $X(u,v) : U \to \mathbb{R}^3$ of a surface S such that the first and second fundamental forms are given by:

   (a) $(g_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $(h_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$;

   (b) $(g_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & \cos^2 u \end{pmatrix}$ and $(h_{ij}) = \begin{pmatrix} \cos^2 u & 0 \\ 0 & 1 \end{pmatrix}$?

   Explain your answer.

2. Prove that if $\alpha : (a,b) \to \mathbb{S}^2$ is a geodesic on the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$, then α lies inside a great circle, i.e. there exists some constant vector $v \in \mathbb{R}^3$ such that $\langle \alpha(t), v \rangle = 0$ for all $t \in (a,b)$.

3. Describe all the geodesics on the right cylinder $S = \{(x,y,z) : x^2 + y^2 = 1\}$.

4. Let p_0 be a pole of a unit sphere $\mathbb{S}^2$ and q, r be two points on the corresponding equator in such a way that the meridians p_0q and p_0r make an angle θ at p_0. Consider a unit vector v tangent to the meridian p_0q at p_0, and take the parallel transport of v along the closed curve made up by the meridian p_0q, the parallel qr, and the meridian rp_0 (Fig. 4-21). Determine the angle between the final position of v after parallel transport with the initial vector v.

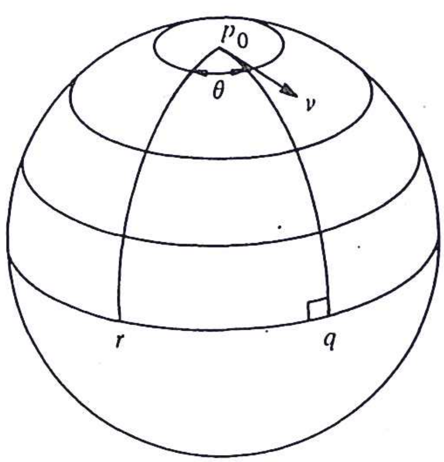

Figure 4-21

Suggested Exercises

(no need to hand in)

1. Show that if all the geodesics of a connected surface are plane curves, then the surface is contained in a plane or a sphere.

2. Let $X(u^1, u^2) : U \to S \subset \mathbb{R}^3$ be a parametrization of a surface. Let ∂_1, ∂_2 be the coordinate vector field $\frac{\partial}{\partial u^1}, \frac{\partial}{\partial u^2}$ respectively and N be its associated unit normal. The first and second fundamental forms are (g_{ij}) and (A_{ij}) and Γ_{ij}^k denotes the Christoffel symbols. Let (g^{ij}) be the inverse matrix of (g_{ij}). The goal of this exercise is to show that the equation

$$\partial_\ell \partial_i N = \partial_i \partial_\ell N \tag{1}$$

simply yields the Codazzi equation.

   (a) Use the Gauss and Weingarten equations to show that

   $$\partial_\ell \partial_i N = [\partial_\ell (g^{kj} A_{ij}) + g^{pj} \Gamma_{\ell p}^k A_{ij}] \partial_k - (g^{pj} A_{\ell p} A_{ij}) N.$$

   Hence, the normal components of (1) are automatically equal.

   (b) Prove that $\partial_\ell g^{kq} = -g^{qi} g^{kp} \partial_\ell g_{pi}$.

   (c) Use (b) and the formula of Γ_{ij}^k in terms of g_{ij} to show that

   $$\partial_\ell g^{kq} + g^{pq} \Gamma_{\ell p}^k = -g^{kj} \Gamma_{j\ell}^q.$$

   (d) Use (a) and (c) to show that the tangential component of (1) gives the Codazzi equation.

3. Study the geodesics on a torus of revolution.