MATH 4030 Differential Geometry
Homework 1
due 13/9/2015 (Tue) at 5PM

Problems

(to be handed in)

Unless otherwise stated, we use I, J to denote connected open intervals in $\mathbb{R}$.

1. Let $\alpha : I \to \mathbb{R}^3$ be a curve and let $\phi : \mathbb{R}^3 \to \mathbb{R}^3$ be a rigid motion. Prove that $L_a^b(\alpha) = L_a^b(\phi \circ \alpha)$. That is, rigid motions preserve the length of curves.

2. Let $\alpha : I \to \mathbb{R}^3$ be a curve and $[a, b] \subset I$. Prove that
$$|\alpha(a) - \alpha(b)| \leq L_a^b(\alpha).$$
In other words, straight lines are the shortest curves joining two given points. *Hint: Use Cauchy-Schwarz inequality.*

3. Let $\phi : J \to I$ be a diffeomorphism between two open intervals $I, J \subset \mathbb{R}$ and let $\alpha : I \to \mathbb{R}^3$ be a curve. Given $[a, b] \subset J$ with $\phi([a, b]) = [c, d]$, prove that $L_a^b(\alpha \circ \phi) = L_c^d(\alpha)$.

Suggested Exercises

(no need to hand in)

1. Find a curve $\alpha : I \to \mathbb{R}^2$ whose trace is the circle $x^2 + y^2 = 1$ such that $\alpha(t)$ runs clockwise around the circle with $\alpha(0) = (0, 1)$.

2. Let $\alpha : I \to \mathbb{R}^3$ br a curve which does not pass through the origin (i.e. $\alpha(t) \neq 0$ for all $t \in I$). If $\alpha(t_0)$ is the point on the trace of α which is closest to the origin and $\alpha'(t_0) \neq 0$, show that the position vector $\alpha(t_0)$ is orthogonal to $\alpha'(t_0)$.

3. Show that if $\alpha : I \to \mathbb{R}^3$ is a curve with $\alpha''(t) = 0$ for all $t \in I$. Show that $\alpha(t)$ is a straight line segment with constant velocity (i.e. $\alpha'(t) =$ a constant vector for all $t \in I$).

4. Let $\alpha : I \to \mathbb{R}^3$ be a curve and $v \in \mathbb{R}^3$ be a fixed vector. Assume that $\langle \alpha'(t), v \rangle = 0$ for all $t \in I$ and that $\langle \alpha(0), v \rangle = 0$. Prove that $\langle \alpha(t), v \rangle = 0$ for all $t \in I$. What does it mean geometrically?

5. Let $\alpha : (-1, +\infty) \to \mathbb{R}^2$ be the *folium of Descartes* given by
$$\alpha(t) = \left(\frac{3t}{1 + t^3}, \frac{3t^2}{1 + t^3} \right).$$
Prove that α is injective but not a homeomorphism onto its image.