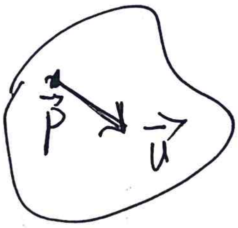
Directional Derivative (review)

rate of change of f along some direction. $\|\vec{u}\| = 1$.



define $D_{\vec{u}} f(\vec{p}) = \lim\limits_{t \to 0} \dfrac{f(\vec{p} + t\vec{u}) - f(\vec{p})}{t}$.

when $\vec{u} = \vec{i}$, $\quad D_{\vec{i}} f(\vec{p}) = \dfrac{\partial f}{\partial x}(\vec{p})$

$\vec{u} = \vec{j}$ $\quad D_{\vec{j}} f(\vec{p}) = \dfrac{\partial f}{\partial y}(\vec{p})$

Thm: If f diff at $\vec{p}$, then when

$\vec{u} = u_1 \vec{i} + u_2 \vec{j}$, we have

$$D_{\vec{u}} f(\vec{p}) = u_1 \frac{\partial f}{\partial x}(\vec{p}) + u_2 \frac{\partial f}{\partial y}(\vec{p})$$

i.e. $D_{\vec{u}} f(\vec{p}) = \nabla f(\vec{p}) \cdot \vec{u}$ $\quad \left(\nabla f = \left(\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}\right)\right)$

when $\vec{u} = (\cos\alpha, \cos\beta, \cos\gamma)$ where

$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$. then if f diff at $\vec{p}$.

we have $D_{\vec{u}} f(\vec{p}) = \frac{\partial f}{\partial x}\Big|_{\vec{p}} \cos\alpha + \frac{\partial f}{\partial y}\Big|_{p} \cos\beta + \frac{\partial f}{\partial z}\Big|_{p} \cos\gamma.$

(3-dim case).

Remark: higher dimension situation is similar

example and exercise:

1. $f: \mathbb{R}^2 \to \mathbb{R}$!

$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

(1) f cts. at $\vec{o}$

(2) $D_{\vec{u}} f(\vec{o})$ exist for all $\vec{u}$

(3) f is not diff. at $\vec{o}$

Sol: omit. see lecture notes.

!remark: for this problem, we don't always have $D_{\vec{u}} f(o) = \left(\frac{\partial f}{\partial x}\Big|_{o}, \frac{\partial f}{\partial y}\Big|_{o}\right) \cdot \vec{u}$

because f is not diff at $\vec{o}$

2. $f(x,y)=\begin{cases} x+y, & x=0 \text{ or } y=0 \\ 1, & \text{others.} \end{cases}$

find $f'_x(0,0)$, $f'_y(0,0)$ and see if the directional derivative exist along the vector ~~$\vec{e}t$~~ $\vec{l}=(a,b)$, $(a\neq 0, b\neq 0)$ at $(0,0)$

sol: $f(0,0)=0$, $f'_x(0,0)=\lim\limits_{x\to 0}\frac{f(x,0)-f(0,0)}{x}$

$=\lim\limits_{x\to 0}\frac{x}{x}=1$. $f'_y(0,0)=1$.

as $D_{\vec{l}}f(0,0)=\lim\limits_{t\to 0}\frac{f(ta,tb)-f(0,0)}{t}=\lim\limits_{t\to 0}\frac{1}{t}=\infty$

$\Rightarrow$ do not exist!

3. $f(x,y)=\begin{cases} \frac{xy^2}{x^2+y^4}, & x\neq 0 \\ 0, & x=0. \end{cases}$, check that the directional derivative at $(0,0)$ along $\vec{\varphi}=(a_1,a_2)$ exist. but $f(x,y)$ ~~doesn't~~ isn't continuous.

sol: $f(0,0)=0$. $\vec{u}=(a_1,a_2)$

$$\lim_{t\to 0}\frac{f(ta_1,ta_2)-f(0,0)}{t}=\lim_{t\to 0}\frac{a_1a_2^2}{a_1^2+t^2a_2^4}$$

$$=\begin{cases} a_2^2/a_1, & a_1\neq 0 \\ 0, & a_1=0 \end{cases} \Rightarrow \text{exist.}$$

but if we set $x=y^2$.

then along $x=y^2$, $f(x,y)=\frac{1}{2}$ $(x,y\neq 0)$

but $f(0,0)=0$. $\Rightarrow$ f not cts.

4. $f(x,y)=\begin{cases} x^3/x^2+y^2, & x^2+y^2\neq 0 \\ 0, & x^2+y^2=0. \end{cases}$

check that ~~f~~ all the directional derivative exist at $(0,0)$. but formula

$$\frac{\partial f}{\partial l}=\nabla f\cdot(\cos\alpha,\cos\beta) \text{ fails sometimes}$$

sol: $f(0,0)=0$

$$f(t\cos\alpha, t\sin\alpha)=\begin{cases} t\cos^3\alpha, & t\neq 0 \\ 0, & t=0 \end{cases}$$

then $$\left.\frac{\partial f}{\partial \ell}\right|_{(0,0)} = \left.\frac{d}{dt} f(t\cos\alpha, t\sin\alpha)\right|_{(0,0)}$$

$$= \cos^3\alpha$$

$$f'_x(0,0) = \lim_{x\to 0} \frac{f(x,0)-f(0,0)}{x} = \frac{x}{x} = 1$$

$$f'_y(0,0) = \lim_{y\to 0} \frac{f(0,y)-f(0,0)}{y} = 0$$

but $\cos^3\alpha$ doesn't always equal $1\cdot\cos\alpha$.