# Differentiability at a give point.

ex1: $f(x,y)=\begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}}, & x^2+y^2\neq 0, \\ 0, & x^2+y^2=0, \end{cases}$ at. $(0,0)$.

$$|f(x,y)| = \left|\frac{xy}{\sqrt{x^2+y^2}}\right| \leq \frac{\sqrt{xy}}{\sqrt{2}} \to 0.$$

$(x,y\to 0)$.

$\Rightarrow f(x,y)$ cts at $(0,0)$

Step 1: (calculate $f'_x(0,0)$, $f'_y(0,0)$)

$$f'_x(0,0) = \lim_{x\to 0} \frac{f(x,0)-f(0,0)}{x} = \lim_{x\to 0} 0 = 0.$$

Partial derivative definition.

$f'_y(0,0)=0$

Step 2: (Check definition of differentiability)

~~f(x,y~~ $\varepsilon(x,y) = f(x,y) - f(0,0) - f'_x(0,0)x - f'_y(0,0)y.$

$$\varepsilon(x,y) = \frac{xy}{\sqrt{x^2+y^2}}$$ ~~but~~

but $\dfrac{\varepsilon(x,y)}{\sqrt{x^2+y^2}} \not\to 0$.

$$\frac{\varepsilon(x,y)}{\sqrt{x^2+y^2}} = \frac{xy}{x^2+y^2}.$$

along $y = kx \Rightarrow \dfrac{\varepsilon(x,y)}{\sqrt{x^2+y^2}} = \dfrac{k}{1+k^2} \to \dfrac{k}{1+k^2}$.

variate with different $k \Rightarrow \lim\limits_{(x,y)\to(0,0)} \dfrac{\varepsilon(x,y)}{\sqrt{x^2+y^2}}$ does not exist.

ex 2: $f(x,y) = \begin{cases} e^{-\frac{1}{x^2+y^2}}, & x^2+y^2 > 0, \\ 0, & x^2+y^2 = 0 \end{cases}$ at $(0,0)$

step 1: $f'_x(0,0) = \lim\limits_{x\to 0} \dfrac{f(x,0)-f(0,0)}{x} = \lim\limits_{x\to 0} x^{-1} e^{-1/x^2} = 0$

$f'_y(0,0) = \lim\limits_{y\to 0} \dfrac{f(0,y)-f(0,0)}{y} = \lim\limits_{y\to 0} y^{-1} e^{-1/y^2} = 0$

step 2: $\varepsilon(x,y) = f(x,y) - f(0,0)$ $\left(f'_x(0,0) = f'_y(0,0) = 0\right)$

$$= e^{-\frac{1}{x^2+y^2}}$$

$$\frac{\varepsilon(x,y)}{\sqrt{x^2+y^2}} = \frac{e^{-\frac{1}{x^2+y^2}}}{\sqrt{x^2+y^2}}$$

$x = r\cos\theta$, $y = r\sin\theta$.

$$\Rightarrow \frac{\varepsilon(x,y)}{\sqrt{x^2+y^2}} = r^{-1} e^{-\frac{1}{r^2}} \to 0 \ (r \to 0).$$

$\Rightarrow f(x,y)$ differentiable at $(0,0)$.

ex3: $f(x,y) = \begin{cases} x^{4/3}\sin(y/x), & x \neq 0 \\ 0, & x = 0, \end{cases}$

at $\forall (x,y) \in \mathbb{R}^2$.

<u>step 1</u>.

$$f_x(x,y) = \begin{cases} \frac{4}{3}x^{\frac{1}{3}}\sin\frac{y}{x} - y x^{-2/3}\cos\frac{y}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

$$f_y(x,y) = \begin{cases} x^{1/3}\cos\frac{y}{x}, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

on $D = \mathbb{R}^2 \setminus \{\{0,y\} : y \in \mathbb{R}^1\}$. $f_x, f_y$ are all cts. $\Rightarrow f(x,y)$ differentiable on $D$.

for $(0,y)$., $f_x(0,y) = f_y(0,y) = 0$

~~$\varepsilon(x,y)$~~

$$\Rightarrow \frac{|f(h,k) - f(0,y)|}{(h^2+(k-y)^2)^{\frac{1}{2}}} \le \frac{\left|h^{\frac{4}{3}} \sin\frac{k}{h}\right|}{|h|}$$

step2

$$\le |h|^{\frac{1}{3}} = o(1) \ (h \to 0)$$

namely $h^{\frac{1}{3}} \to 0$.

$$\Rightarrow \frac{|f(h,k) - f(0,y)|}{(h^2+(k-y)^2)^{\frac{1}{2}}} \to 0. \ (h \to 0, k \to y)$$

$\Rightarrow f$ diff at $(0,y)$.

$\Rightarrow f$ diff at all $(x,y) \in \mathbb{R}^2$

ex4:



$$f(x,y)=\begin{cases}(x^2+y^2)\sin\left(\frac{1}{x^2+y^2}\right), & x^2+y^2\neq 0\\ 0, & x^2+y^2=0.\end{cases}$$ at $(0,0)$

step1:

when $\cancel{\phantom{x}}$ $x^2+y^2\neq 0$

$$f_x'(x,y)=2x\sin\left(\frac{1}{x^2+y^2}\right)-\frac{2x}{x^2+y^2}\cos\frac{1}{x^2+y^2}$$

$$f_y'(x,y)=2y\sin\left(\frac{1}{x^2+y^2}\right)-\frac{2y}{x^2+y^2}\cos\frac{1}{x^2+y^2}$$

at $x^2+y^2=0$.

$$f_x'(0,0)=\lim_{x\to 0}\frac{f(x,0)-f(0,0)}{x}=\lim_{x\to 0}\frac{x^2\sin\frac{1}{x^2}}{x}=0.$$

$$f_y'(0,0)=0$$

step2: $\underbrace{f(x,y)-f(0,0)}_{\varepsilon(x,y)=}=(x^2+y^2)\sin\frac{1}{x^2+y^2}$

$$\frac{\varepsilon(x,y)}{\sqrt{x^2+y^2}}=\sqrt{x^2+y^2}\sin\frac{1}{x^2+y^2}\to 0$$ by poler coodinate.

$\Rightarrow f$ diff at $(0,0)$

# Chain Rule.

idea: from outer level to inner level.

ex1: $f = \ln(x^2+y^2)$

$$\frac{\partial f}{\partial x} = \frac{1}{x^2+y^2} \cdot 2x \quad \text{(treat } y \text{ as constant)}$$

$$\frac{\partial f}{\partial y} = \frac{1}{x^2+y^2} \cdot 2y \quad (x \text{ constant})$$

ex2: $f = \ln\sqrt{x^2+y^2} \quad (x^2+y^2>0)$

$$\frac{\partial f}{\partial x} = \frac{1}{\sqrt{x^2+y^2}} \cdot \frac{1}{2}(x^2+y^2)^{-\frac{1}{2}} \cdot 2x$$

$$\cancel{\frac{\partial f}{\partial y}} = \cancel{\frac{1}{\sqrt{x^2+y^2}}} \qquad\qquad = \frac{x}{x^2+y^2}$$

$$\frac{\partial f}{\partial y} = \frac{y}{x^2+y^2}$$