MATH 2010B Advanced Calculus I
(2014-2015, First Term)
Quiz 3
Suggested Solution

Question 1.

$$f(x,y) = \begin{cases} \dfrac{2xy^2}{x^2+y^4} & \text{when } (x,y) \neq (0,0) \\ 0 & \text{when } (x,y) = (0,0) \end{cases}$$

(a) $\vec{u} = (\cos\theta, \sin\theta)$.

Case 1: $\cos\theta \neq 0$,

$$\begin{aligned} D_{\vec{u}}f(0,0) &= \lim_{h\to 0} \frac{f(0+h\cos\theta, 0+h\sin\theta) - f(0,0)}{h} \\ &= \lim_{h\to 0} \frac{\frac{2h^3\cos\theta\sin^2\theta}{h^2\cos^2\theta + h^4\sin^4\theta}}{h} \\ &= \lim_{h\to 0} \frac{2\cos\theta\sin^2\theta}{\cos^2\theta + h^2\sin^4\theta} \\ &= \frac{2\cos\theta\sin^2\theta}{\cos^2\theta} \\ &= 2\sin\theta\tan\theta \end{aligned}$$

Case 2: $\cos\theta = 0$, then $\vec{u} = (0,1)$,

$$\begin{aligned} D_{\vec{u}}f(0,0) &= \lim_{h\to 0} \frac{f(0,0+h) - f(0,0)}{h} \\ &= \lim_{h\to 0} \frac{\frac{2(0)^2(h)^2}{(0)^2+(h)^4}}{h} \\ &= 0 \end{aligned}$$

(b) Along $x = my^2$, we have

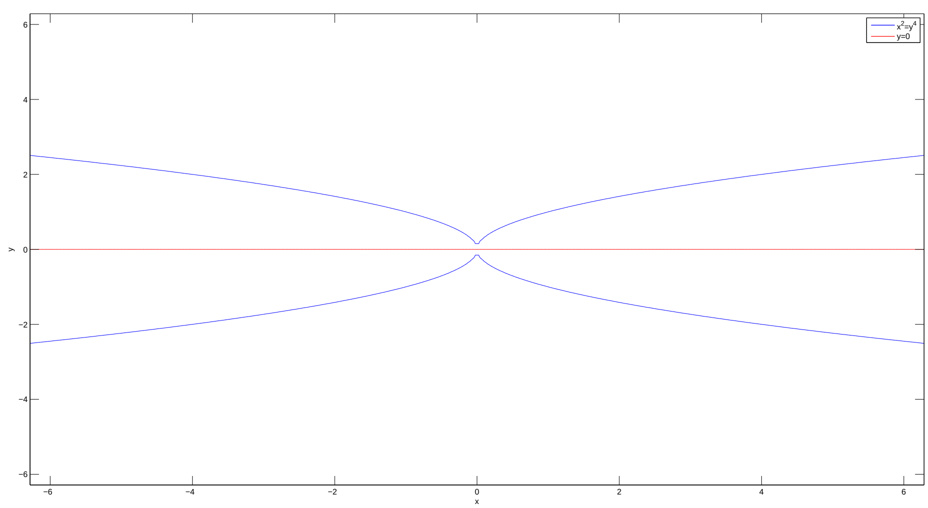
$$\begin{aligned} &\lim_{(x,y)\to(0,0)} f(x,y) \\ =& \lim_{y\to 0} \frac{2my^4}{m^2y^4 + y^4} \\ =& \lim_{y\to 0} \frac{2m}{m^2+1} \\ =& \frac{2m}{m^2+1} \end{aligned}$$

which has different value for different m. Thus f is not continuous at $(0,0)$. Therefore, f is not differentiable at $(0,0)$.

(c) For $(x, y) = (0, 0)$, from the calculation in (a), we have $\nabla f = (0, 0)$.
For $(x, y) \neq (0, 0)$,

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{2xy^2}{x^2+y^4}\right) \\ &= \frac{(2y^2)(x^2+y^4)-(2xy^2)(2x)}{(x^2+y^4)^2} \\ &= \frac{-2y^2(x^2-y^4)}{(x^2+y^4)^2} \\ \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{2xy^2}{x^2+y^4}\right) \\ &= \frac{(4xy)(x^2+y^4)-(2xy^2)(4y^3)}{(x^2+y^4)^2} \\ &= \frac{4xy(x^2-y^4)}{(x^2+y^4)^2}\end{aligned}$$

Set $f_x = f_y = 0$, since $(x, y) \neq (0, 0)$, we get $x^2 - y^4 = 0 \Rightarrow x^2 = y^4$ and $y = 0$.
Therefore, $S = \{(x, y) \in \mathbb{R}^2 : \nabla f(x, y) = 0\} = \{(x, y) \in \mathbb{R}^2 : x^2 = y^4 \text{ or } y = 0\}$. The graph of the set is as follows



(d) $\gamma(0) = (1, 1)$. Note that $(1, 1)$ belongs to the set S in question (c). Therefore, $f_x(\gamma(0)) = f_x(1, 1) = 0$ and $f_y(\gamma(0)) = f_y(1, 1) = 0$.
$\gamma'(t) = (-2\cos t \sin t, 100(1+t)^{99})$, $\gamma'(0) = (0, 100)$.
Then

$$\frac{\partial}{\partial t}\Big|_{t=0} f(\gamma(t)) = (f_x(1, 1), f_y(1, 1)) \cdot \gamma'(0) = 0$$

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