MATH 2010A/B Advanced Calculus I
(2014-2015, First Term)
Homework 7
Suggested Solution

1. (a) $f(x,y) = e^x \cos y$ at $(0, \pi/2)$.
$f_x = e^x \cos y$ and $f_y = -e^x \sin y$.
Then at $(0, \pi/2)$, $f = 0, f_x = 0$ and $f_y = -1$.
Therefore,

$$L(x,y) = (0) + (0)(x-0) + (-1)(y-\pi/2) = -y + \pi/2$$

(b) $f(x,y) = x^3y^4$ at $(1,1)$.
$f_x = 3x^2y^4$ and $f_y = 4x^3y^3$.
Then at $(1,1)$, $f = 1, f_x = 3$ and $f_y = 4$.
Therefore,

$$L(x,y) = (1) + (3)(x-1) + (4)(y-1) = 3x + 4y - 6$$

2. (a) $f(x,y,z) = \sqrt{x^2+y^2+z^2}$ at $(1,1,0)$.
$f_x = x(x^2+y^2+z^2)^{-\frac{1}{2}}$, $f_y = y(x^2+y^2+z^2)^{-\frac{1}{2}}$, $f_z = z(x^2+y^2+z^2)^{-\frac{1}{2}}$.
Then at $(1,1,0)$, $f = \sqrt{2}, f_x = \frac{1}{\sqrt{2}}$, $f_y = \frac{1}{\sqrt{2}}$ and $f_z = 0$.
Therefore,

$$L(x,y,z) = (\sqrt{2}) + \left(\frac{1}{\sqrt{2}}\right)(x-1) + \left(\frac{1}{\sqrt{2}}\right)(y-1) + (0)(z-1) = \frac{\sqrt{2}x}{2} + \frac{\sqrt{2}y}{2}$$

(b) $f(x,y,z) = e^x + \cos(y+z)$ at $(0, \frac{\pi}{2}, 0)$.
$f_x = e^x$, $f_y = -\sin(y+z)$, $f_z = -\sin(y+z)$.
Then at $(0, \frac{\pi}{2}, 0)$, $f = 1, f_x = 1$, $f_y = -1$ and $f_z = -1$.
Therefore,

$$L(x,y,z) = (1) + (1)(x-0) + (-1)(y - \frac{\pi}{2}) + (-1)(z-0) = x - y - z + 1 + \frac{\pi}{2}$$

3.

$$f(x,y) = \begin{cases} \dfrac{xy^2}{x^2+y^4}, & (x,y) \neq (0,0) \\ 1, & (x,y) = (0,0) \end{cases}$$

Then

$$\begin{aligned} f_x(0,0) &= \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h} \\ &= \lim_{h\to 0} \frac{\frac{(h)(0)^2}{h^2+0^4} - 0}{h} \\ &= \lim_{h\to 0} \frac{0-0}{h} \\ &= 0 \end{aligned}$$

And

$$
\begin{aligned}
f_y(0,0) &= \lim_{h \to 0} \frac{f(0,0+h)-f(0,0)}{h} \\
&= \lim_{h \to 0} \frac{\frac{(0)(h)^2}{0^2+h^4}-0}{h} \\
&= 0
\end{aligned}
$$

But along the curve $x = y^2$, we have

$$
\begin{aligned}
\lim_{(x,y)\to(0,0)} f(x,y) &= \lim_{(x,y)\to(0,0)} \frac{y^4}{y^4+y^4} \\
&= \lim_{(x,y)\to(0,0)} \frac{y^4}{2y^4} \\
&= \frac{1}{2} \\
&\neq f(0,0)
\end{aligned}
$$

Therefore, f is not continuous at $(0,0) \Rightarrow f$ is not differentiable at $(0,0)$.

4.

$$
f(x,y) = \begin{cases} 0, & x^2 < y < 2x^2 \\ 1, & \text{otherwise} \end{cases}
$$

Then

$$
\begin{aligned}
f_x(0,0) &= \lim_{h \to 0} \frac{f(0+h,0)-f(0,0)}{h} \\
&= \lim_{h \to 0} \frac{1-1}{h} \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
f_y(0,0) &= \lim_{h \to 0} \frac{f(0,0+h)-f(0,0)}{h} \\
&= \lim_{h \to 0} \frac{1-1}{h} \\
&= 0
\end{aligned}
$$

But along the curve $x^2 < y = \frac{3}{2}x^2 < 2x^2$, we have

$$
\begin{aligned}
\lim_{(x,y)\to(0,0)} f(x,y) &= \lim_{(x,y)\to(0,0)} 0 \\
&= 0 \\
&\neq f(0,0)
\end{aligned}
$$

Therefore, f is not continuous at $(0,0) \Rightarrow f$ is not differentiable at $(0,0)$.

5.

$$f(x,y) = \begin{cases} y^2 + x^2 \sin \dfrac{1}{x}, & x \neq 0 \\ y^2, & x = 0 \end{cases}$$

First compute

$$\begin{aligned} f_x(0,0) &= \lim_{h \to 0} \frac{f(0+h,0) - f(0,0)}{h} \\ &= \lim_{h \to 0} \frac{0^2 + h^2 \sin \frac{1}{h} - 0}{h} \\ &= \lim_{h \to 0} h \sin \frac{1}{h} \\ &= 0 \end{aligned}$$

Since $\left| \sin \frac{1}{h} \right|$ is bounded. And

$$\begin{aligned} f_y(0,0) &= \lim_{h \to 0} \frac{f(0,0+h) - f(0,0)}{h} \\ &= \lim_{h \to 0} \frac{h^2 - 0}{h} \\ &= 0 \end{aligned}$$

Then we show that

$$\begin{aligned} &\lim_{(h,k) \to (0,0)} \frac{f(0+h,0+k) - [f(0,0) + (0)(h) + (0)(k)]}{\sqrt{h^2+k^2}} \\ = &\lim_{(h,k) \to (0,0)} \frac{f(h,k)}{\sqrt{h^2+k^2}} \\ = &\lim_{(h,k) \to (0,0)} \frac{k^2 + h^2 \sin \frac{1}{h}}{\sqrt{h^2+k^2}} \end{aligned}$$

Take $h = r\cos\theta$ and $k = r\sin\theta$, we have

$$\begin{aligned} &\lim_{(h,k) \to (0,0)} \frac{k^2 + h^2 \sin \frac{1}{h}}{\sqrt{h^2+k^2}} \\ = &\lim_{r \to 0} \frac{r^2(\sin^2 + \cos^2) \sin \frac{1}{r\cos\theta}}{r} \\ = &\lim_{r \to 0} r(\sin^2 + \cos^2) \sin \frac{1}{r\cos\theta} \\ = &\ 0 \end{aligned}$$

Since $\left| (\sin^2 + \cos^2) \sin \frac{1}{r\cos\theta} \right|$ is bounded. Therefore, f is differentiable at $(0,0)$.

Note that for $x \neq 0$, $f_x = 2x \sin \frac{1}{x} - \cos \frac{1}{x}$. Then

$$\lim_{(x,y)\to(0,0)} f_x \quad = \quad \lim_{(x,y)\to(0,0)} 2x \sin \frac{1}{x} - \cos \frac{1}{x}$$

does not exist since $\lim_{x\to 0} \cos \frac{1}{x}$ does not exist. Therefore, f_x is not continuous at $(0, 0)$. Thus, f is not continuous differentiable in any neighbourhood of $(0, 0)$.

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