MATH 4030 Differential Geometry
Homework 6
Suggested solutions

1. (2 points) The idea is to show that a surface S with this property has to be totally umbilic (i.e. the shape operator $\mathcal{S}_p : T_pS \to T_pS$ is a scalar multiple of the identity), and thus by Lecture notes (part 4) p.15-16 S is contained in a plane or a sphere. Let $p \in S$ be any point, and let $v \in T_pS$ be any tangent vector. We need to show that $\mathcal{S}_p$ takes v to a multiple of itself. It suffices to assume $|v| = 1$. We know that there is a (unique) geodesic $\alpha : (-\varepsilon, \varepsilon) \to S$ with $\alpha(0) = p$ and $\alpha'(0) = v$. Since $|a'|$ is constant, it follows that α is p.b.a.l.. Recall that
$$\alpha'' = D_{\alpha'}\alpha' = A(\alpha', \alpha')N \circ \alpha + \nabla_{\alpha'}\alpha'$$
where N is the normal of S. To avoid confusion, we let N_α be the normal of α as a space curve. Since α is a geodesic, $\nabla_{\alpha'}\alpha' = 0$, and hence α'' is parallel to N. In other words, N_α is equal to $\pm N \circ \alpha$. Since we have no preference for the sign of N, we may assume $N_\alpha = N \circ \alpha$.

   By the given assumption, α is a plane curve. Recall that α is a plane curve if and only if the torsion τ of α is identically zero, but attention this is true only when the curvature κ of α is non-zero everywhere (otherwise τ is not even well-defined). Let us first assume that $\alpha''(0) \neq 0$ which implies $\kappa \neq 0$ near 0 so that τ is well-defined there. We will address the case $\alpha''(0) = 0$ later. Under this assumption, we have $\tau(0) = 0$, and so $N_\alpha'(0) = -\kappa(0)\alpha'(0)$. It follows that
$$\mathcal{S}_p(v) = -dN_p(v) = -\frac{d}{ds}(N \circ \alpha)\bigg|_{s=0} = -N_\alpha'(0) = \kappa(0)\alpha'(0) = \kappa(0)v.$$
We have shown that if the geodesic α_v corresponding to v satisfies $\alpha_v''(0) \neq 0$, then v is an eigenvector of $\mathcal{S}_p$. Now what if $\alpha''(0) = 0$? Well, we don't need to deal with it because in order to show $\mathcal{S}_p$ is a scalar multiple of the identity, one only needs to check that $\mathcal{S}_p$ has at least 5 unit eigenvectors (why?). If it happens that we cannot find 5 distinct unit vectors $v \in T_pS$ such that the geodesic α_v defined above satisfies $\alpha_v''(0) \neq 0$, then the second fundamental form $A_p(v, v)$ will vanish for almost all $v \in T_pS$, since $\alpha_v''(0) = A_p(v, v)N_p$. But then $A_p(v, v)$ will vanish for all v, and hence $\langle v, \mathcal{S}_p w\rangle = A_p(v, w) = 0$ for all $v, w \in T_pS$, giving $\mathcal{S}_p \equiv 0$ in which case $\mathcal{S}_p$ is also a scalar multiple of the identity.

2. (2 points) Suppose there are two simple closed geodesics α_1 and α_2 on S which do not intersect. By Q3 below, we know that S is homeomorphic to the sphere, and hence by Jordan curve theorem, α_1 separates S into two regions D_1 and D_2, each homeomorphic to the disk. WLOG, assume D_1 contains α_2 (entirely, since $\alpha_1 \cap \alpha_2 = \emptyset$). Then by Jordan curve theorem again, α_2 bounds a disk D_3 which lies entirely in D_1. Now it is easy to see that α_1 and α_2 together bound a cylinder Σ.

   By applying Gauss-Bonnet theorem to Σ, we have
$$\int_\Sigma K dA \ \pm \int_{\alpha_1} \kappa_g ds \ \pm \int_{\alpha_2} \kappa_g ds = 2\pi\chi(\Sigma).$$
Since α_1 and α_2 are geodesics, and $\chi(\Sigma) = \chi(\text{circle}) = 0$, it follows that $\int_\Sigma K dA = 0$, in contradiction to the assumption that $K > 0$ everywhere. Therefore, any two simple closed geodesics on S must intersect.

3. (1 point) (Revised: Σ is connected.) First by HW4 Q1, we know that Σ contains a point p such that $K(p) > 0$. Next by the classification of closed orientable surfaces, we have $\chi(\Sigma) = 2 - 2g$ where g is the genus of Σ which is greater than 0 if Σ is not homeomorphic to the sphere. Hence $\chi(\Sigma) \leqslant 0$. Then by Gauss-Bonnet theorem $\int_\Sigma K dA = 2\pi\chi(\Sigma) \leqslant 0$, we see that K cannot be non-negative everywhere (since K is positive somewhere), in other words, there is $q \in \Sigma$ such that $K(q) < 0$. Finally by intermediate value theorem, Σ contains a point r (lying in each path in Σ joining p and q) such that $K(r) = 0$.

4. (1 point) By HW4 Q4, we know that the area element dA of the given torus T is

$$dA = |X_u \times X_v| \; dudv = (2 + \cos u) dudv$$

and the Gauss curvature K of T is

$$K = \frac{\cos u}{(2 + \cos u)}$$

where we have put $a = 2$ and $b = 1$. It follows that

$$\begin{aligned}\int_T K dA &= \int_0^{2\pi} \int_0^{2\pi} \frac{\cos u}{(2 + \cos u)} \cdot (2 + \cos u) dudv \\ &= \int_0^{2\pi} \int_0^{2\pi} \cos u \; dudv \\ &= 0.\end{aligned}$$

5. (2 points)

(a) We have

$$\begin{aligned}\beta'(s) &= \alpha'(s) - rN'(s) \\ &= \alpha'(s) - rJ\alpha''(s) \\ &= \alpha'(s) - rJ(\kappa(s)N(s)) \\ &= (1 + r\kappa(s))\alpha'(s)\end{aligned}$$

(where J denotes the rotation by $90°$ anti-clockwise as usual). It follows that

$$\text{Length}(\beta) = \int_0^L |\beta'(s)| \; ds = \int_0^L (1 + r\kappa(s)) ds = L + 2\pi r = \text{Length}(\alpha) + 2\pi r.$$

(Here we have used the Gauss-Bonnet theorem for plane curves and the convexity of α which implies $\kappa \geqslant 0$.)

(b) By Green's formula, we have

$$\begin{aligned}
\text{Area}(\Omega_\beta) &= -\frac{1}{2}\int_0^L \langle \beta(s), J\beta'(s)\rangle \, ds \\
&= -\frac{1}{2}\int_0^L \langle \alpha(s) - rN(s), (1+r\kappa(s))J\alpha'(s)\rangle \, ds \\
&= -\frac{1}{2}\left[\int_0^L \langle \alpha(s), J\alpha'(s)\rangle \, ds + \int_0^L \langle \alpha(s), r\kappa(s)N(s)\rangle \, ds - \int_0^L r(1+r\kappa(s))\langle N(s), N(s)\rangle \, ds\right] \\
&= \text{Area}(\Omega_\alpha) - \frac{1}{2}\int_0^L r\langle \alpha(s), \alpha''(s)\rangle \, ds + \frac{1}{2}\int_0^L r(1+r\kappa(s)) \, ds \\
&= \text{Area}(\Omega_\alpha) + \frac{1}{2}\int_0^L r\langle \alpha'(s), \alpha'(s)\rangle \, ds + \frac{1}{2}(rL + 2\pi r^2) \\
&= \text{Area}(\Omega_\alpha) + \frac{1}{2}rL + \frac{1}{2}rL + \pi r^2 \\
&= \text{Area}(\Omega_\alpha) + rL + \pi r^2.
\end{aligned}$$

6. (2 points)

(a) This follows from $N' = -\kappa T - \tau B$.

(b)

$$\frac{dN}{dt} = \frac{dN}{ds}\cdot\frac{ds}{dt} = \frac{-\kappa T - \tau B}{|N'(s)|} = \frac{1}{\sqrt{\kappa^2+\tau^2}}(-\kappa T - \tau B)$$

(c) First notice that the normal $\mathbf{n}$ of $N(t)$ in $\mathbb{S}^2$ is

$$\mathbf{n}(t) = N(t) \times N'(t) = \frac{1}{\sqrt{\kappa^2+\tau^2}}(-\tau T + \kappa B).$$

Then we have

$$\begin{aligned}
\kappa_g &= \langle N''(t), \mathbf{n}\rangle \\
&= \frac{1}{\sqrt{\kappa^2+\tau^2}}\left[-\tau\left\langle \frac{d}{ds}\left(\frac{1}{\sqrt{\kappa^2+\tau^2}}(-\kappa T - \tau B)\right), T\right\rangle + \kappa\left\langle \frac{d}{ds}\left(\frac{1}{\sqrt{\kappa^2+\tau^2}}(-\kappa T - \tau B)\right), B\right\rangle\right]\left(\frac{dt}{ds}\right)^{-1} \\
&= \frac{1}{\sqrt{\kappa^2+\tau^2}}\left[-\tau\frac{d}{ds}\left\langle\left(\frac{1}{\sqrt{\kappa^2+\tau^2}}(-\kappa T - \tau B)\right), T\right\rangle + \kappa\frac{d}{ds}\left\langle\left(\frac{1}{\sqrt{\kappa^2+\tau^2}}(-\kappa T - \tau B)\right), B\right\rangle\right]\left(\frac{dt}{ds}\right)^{-1} \\
&= \frac{1}{\sqrt{\kappa^2+\tau^2}}\left[-\tau\frac{d}{ds}\left(\frac{-\kappa}{\sqrt{\kappa^2+\tau^2}}\right) + \kappa\frac{d}{ds}\left(\frac{-\tau}{\sqrt{\kappa^2+\tau^2}}\right)\right]\left(\frac{dt}{ds}\right)^{-1} \\
&= -\left(\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}\right)^2\frac{d}{ds}\left(\frac{\tau}{\sqrt{\kappa^2+\tau^2}}\Big/\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}\right)\left(\frac{dt}{ds}\right)^{-1} \\
&= -\frac{1}{1+(\frac{\tau}{\kappa})^2}\frac{d}{ds}\left(\frac{\tau}{\kappa}\right)\left(\frac{dt}{ds}\right)^{-1} \\
&= -\frac{d}{ds}\left(\tan^{-1}\frac{\tau}{\kappa}\right)\left(\frac{dt}{ds}\right)^{-1}.
\end{aligned}$$

(d) By Gauss-Bonnet theorem, it suffices to show that the integral of the geodesic curvature of N along N is zero. Indeed,

$$\begin{aligned}
\int_N \kappa_g dt &= \int_0^L -\frac{d}{ds}\left(\tan^{-1}\frac{\tau}{\kappa}\right)\left(\frac{dt}{ds}\right)^{-1} \, dt \\
&= -\int_0^L \frac{d}{ds}\left(\tan^{-1}\frac{\tau}{\kappa}\right) \, ds \\
&= 0 \quad (\because N \text{ is a simple closed curve}).
\end{aligned}$$