MATH 4030 Differential Geometry
Homework 4
Suggested solutions

1. (2 points)

(a) Let S be a surface and $p_0 \notin S$. From Tut7, we know that the Hessian of the distance-square function $f(x) = |x - p_0|^2$ at a critical point $p \in S$ and the second fundamental form A_N of S at p with respect to the normal $N = \frac{p_0 - p}{|p_0 - p|}$ are related by the formula

$$\mathrm{Hess}(f)_p(\zeta, \theta) = 2[\langle \zeta, \theta \rangle - |p_0 - p|(A_N)_p(\zeta, \theta)]$$

for any tangent vectors $\zeta, \theta \in T_pS$. See also HW4 suggested ex. Q1.

Recall also what the Hessian of f can tell us about the (local) behaviour of f near the critical point p:

- $\mathrm{Hess}(f)_p > 0 \Longrightarrow f$ attains local minimum at p;
- $\mathrm{Hess}(f)_p < 0 \Longrightarrow f$ attains local maximum at p;
- f attains local minimum at $p \Longrightarrow \mathrm{Hess}(f)_p \geqslant 0$;
- f attains local maximum at $p \Longrightarrow \mathrm{Hess}(f)_p \leqslant 0$.

($\Longleftarrow$) Suppose p is a local maximum of f. Then we have $2[\langle \zeta, \zeta \rangle - |p_0 - p|(A_N)_p(\zeta, \zeta)] = \mathrm{Hess}(f)_p \leqslant 0$ for any $\zeta \in T_pS$. This shows that $(A_N)_p > 0$. Since the sign of the Gauss curvature K is equal to the sign of the determinant of the matrix representing A_N with respect to the basis $\{X_u, X_v\}$ coming from any chart (that's because $K = \frac{\det(A)}{\det(g)}$), it follows that $K(p) > 0$.

($\Longrightarrow$) Given $K(p) > 0$. Choose the unit normal N at p such that one of the (and hence both, by the assumption) principal curvatures κ_1 and κ_2 at p are positive. Choose a positive real number R such that $R \cdot \min(\kappa_1, \kappa_2) > 1$ (warning: the strict inequality sign cannot be replaced by the equality sign, see Remark 3). Let $p_0 = p + R \cdot N$. It follows that

- p is a critical point of the function $f(x) = |x - p_0|^2$;
- $(A_N)_p(\zeta, \zeta) \geqslant \min(\kappa_1, \kappa_2)|\zeta|^2$ (see Remark 2) so that

$$\mathrm{Hess}(f)_p(\zeta, \zeta) \leqslant 2\left(1 - R \cdot \min(\kappa_1, \kappa_2)\right)|\zeta|^2 < 0$$

for any non-zero $\zeta \in T_pS$.

In other words, $\mathrm{Hess}(f)_p < 0$ which implies that f attains local maximum at p.

(b) This is true because (a) shows that every compact surface S in $\mathbb{R}^3$ has at least one point at which the Gauss curvature is strictly positive: simply choose a point p_0 outside S, then the point at which $f(x) = |x - p_0|^2$ attains maximum is the desired point.

Remark 1. (b) implies that in $\mathbb{R}^3$ there is no compact minimal surface (without boundary) as $H = 0 \implies K \leqslant 0$.

Remark 2. The second fundamental form of S at p with respect to N is defined to be either the matrix $\begin{pmatrix} \langle X_{uu}, N \rangle & \langle X_{uv}, N \rangle \\ \langle X_{uv}, N \rangle & \langle X_{vv}, N \rangle \end{pmatrix}$ using a chart X, or the symmetric bilinear form A_N defined by

$$A_N(\zeta, \theta) = \langle \zeta, \mathcal{S}\theta \rangle$$

where $\mathcal{S}$ is the shape operator with respect to N. From this we see that if κ_1, κ_2 are the principal curvatures (= eigenvalues of $\mathcal{S}$) and ζ_1, ζ_2 are two corresponding principal directions (= unit eigenvectors of $\mathcal{S}$), then by expressing any vector ζ by $a\zeta_1 + b\zeta_2$, we have

$$A_N(\zeta, \zeta) = \kappa_1 a^2 + \kappa_2 b^2 \geqslant \min(\kappa_1, \kappa_2)|\zeta|^2.$$

Remark 3. We cannot take R such that $R \cdot \min(\kappa_1, \kappa_2) = 1$. Here is a counter-example: take S to be the graphical surface

$$\left\{ \left(x, y,\ g(x,y) = -1 + \frac{1}{2}(x^2 + y^2) + x^3 \right) \mid x, y \in \mathbb{R} \right\}$$

and $p = (0, 0, -1)$. Then the first and the second fundamental form with respect to the chart $X : (u, v) \mapsto (u, v, g(u, v))$ and the upward normal are both equal to the identity matrix so that $\kappa_1 = \kappa_2 = 1$. If we take $R = \frac{1}{\min(\kappa_1, \kappa_2)} = 1$, then $p_0 = (0, 0, 0)$, and so the corresponding distance-square function f reads

$$(f \circ X)(u, v) = u^2 + v^2 + \left(-1 + \frac{1}{2}(u^2 + v^2) + u^3 \right)^2.$$

Now put $v = 0$, so we have $(f \circ X)(u, 0) = 1 - 2u^3 + \frac{1}{4}u^4 + u^5 + u^6$. Because the leading order term following the constant is $-2u^3$ which is an odd function, we conclude that f does not attain local maximum at p.

2. (1 point)

Catenoid:

$$\begin{aligned}
X_u &= (-\cosh v \sin u, \cosh v \cos u, 0) \\
X_v &= (\sinh v \cos u, \sinh v \sin u, 1) \\
g &= \begin{pmatrix} \cosh^2 v & 0 \\ 0 & \cosh^2 v \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
X_u \times X_v &= (\cosh v \cos u, \cosh v \sin u, -\cosh v \sinh v) \\
N = \frac{X_u \times X_v}{|X_u \times X_v|} &= \frac{1}{\cosh v}(\cos u, \sin u, -\sinh v)
\end{aligned}$$

$$\begin{aligned}
\mathcal{S}(X_u) &= -\frac{\partial N}{\partial u} \\
&= \frac{-1}{\cosh v}(-\sin u, \cos u, 0) \\
&= -\frac{1}{\cosh^2 v} \cdot X_u + 0 \cdot X_v
\end{aligned}$$

$$\begin{aligned}
\mathcal{S}(X_v) &= -\frac{\partial N}{\partial v} \\
&= \frac{\sinh v}{\cosh^2 v}\left(\cos u, \sin u, -\sinh v + \frac{\cosh^2 v}{\sinh v} \right) \\
&= 0 \cdot X_u + \frac{1}{\cosh^2 v} \cdot X_v
\end{aligned}$$

$$[\mathcal{S}]_{\{X_u,X_v\}} = \begin{pmatrix} \frac{-1}{\cosh^2 v} & 0 \\ 0 & \frac{1}{\cosh^2 v} \end{pmatrix}$$

$$H = 0 \qquad K = \frac{-1}{\cosh^4 v}$$

Helicoid:

$$X_u = (-v\sin u, v\cos u, 1)$$
$$X_v = (\cos u, \sin u, 0)$$
$$g = \begin{pmatrix} 1+v^2 & 0 \\ 0 & 1 \end{pmatrix}$$

$$X_u \times X_v = (-\sin u, \cos u, -v)$$
$$N = \frac{X_u \times X_v}{|X_u \times X_v|} = \frac{1}{\sqrt{1+v^2}}(-\sin u, \cos u, -v)$$

$$\begin{aligned} \mathcal{S}(X_u) &= -\frac{\partial N}{\partial u} \\ &= \frac{1}{\sqrt{1+v^2}}(\cos u, \sin u, 0) \\ &= 0 \cdot X_u + \frac{1}{\sqrt{1+v^2}} \cdot X_v \end{aligned}$$

$$\begin{aligned} \mathcal{S}(X_v) &= -\frac{\partial N}{\partial v} \\ &= \frac{1}{\sqrt{1+v^2}^3}(-v\sin u, v\cos u, 1) \\ &= \frac{1}{\sqrt{1+v^2}^3} \cdot X_u + 0 \cdot X_v \end{aligned}$$

$$[\mathcal{S}]_{\{X_u,X_v\}} = \begin{pmatrix} 0 & \frac{1}{\sqrt{1+v^2}^3} \\ \frac{1}{\sqrt{1+v^2}} & 0 \end{pmatrix}$$

$$H = 0 \qquad K = -\frac{1}{(1+v^2)^2}$$

3. (1 point) From HW3 Q7, we know that the mean curvature H of a graphical surface $\{(x, y, f(x, y))\}$ is given by

$$\frac{(1+f_v^2)f_{uu} - 2f_uf_vf_{uv} + (1+f_u^2)f_{vv}}{(f_u^2 + f_v^2 + 1)^{\frac{3}{2}}}.$$

It follows that the surface is minimal if and only if the numerator

$$(1+f_v^2)f_{uu} - 2f_uf_vf_{uv} + (1+f_u^2)f_{vv}$$

is identically zero.

4. (1 point)

$$
\begin{aligned}
X_u &= (-b \sin u \cos v, -b \sin u \sin v, b \cos u) \\
X_v &= (-(a + b\cos u)\sin v, (a + b\cos u)\cos v, 0) \\
g &= \begin{pmatrix} b^2 & 0 \\ 0 & (a + b\cos u)^2 \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
X_u \times X_v &= b(a + b\cos u)(-\cos u \cos v, -\cos u \sin v, -\sin u) \\
N = \frac{X_u \times X_v}{|X_u \times X_v|} &= (-\cos u \cos v, -\cos u \sin v, -\sin u)
\end{aligned}
$$

$$
\begin{aligned}
X_{uu} &= (-b \cos u \cos v, -b \cos u \sin v, -b \sin u) \\
X_{uv} &= (b \sin u \sin v, -b \sin u \cos v, 0) \\
X_{vv} &= (-(a + b\cos u)\cos v, -(a + b\cos u)\sin v, 0) \\
A &= \begin{pmatrix} b & 0 \\ 0 & (a + b\cos u)\cos u \end{pmatrix}
\end{aligned}
$$

$$
[\mathcal{S}]_{\{X_u, X_v\}} = g^{-1}A = \begin{pmatrix} \frac{1}{b} & 0 \\ 0 & \frac{\cos u}{a + b\cos u} \end{pmatrix}
$$

$$
H = \frac{1}{b} + \frac{\cos u}{a + b\cos u} \qquad K = \frac{\cos u}{b(a + b\cos u)}
$$

5. (1 point) First note that $f(S_1) \subseteq S_2$ so that it is a well-defined smooth map $f : S_1 \to S_2$. Next we want to show that the first fundamental form of S_1 (with respect to the chart $X : (u, v) \mapsto (u, v, 0)$) is the same as the one "pull-backed" from S_2, namely the matrix obtained by taking the inner products of the vectors $df(X_u)$ and $df(X_v)$. We have

$$
\begin{aligned}
df(X_u) &= \frac{\partial(f \circ X)}{\partial u} = (-\sin u, \cos u, 0) \\
df(X_v) &= \frac{\partial(f \circ X)}{\partial v} = (0, 0, 1) \\
\langle df(X_u), df(X_u) \rangle &= 1 \\
\langle df(X_u), df(X_v) \rangle &= 0 \\
\langle df(X_v), df(X_v) \rangle &= 1
\end{aligned}
$$

so that the "pull-back" of the first fundamental form of S_2 is the identity matrix which is the same as the first fundamental form of S_1 with respect to the chart X.

6. (2 points) Let $X : \mathbb{R}_{>0} \times (0, \pi) \to S_1 : (r, \theta) \mapsto (r\cos\theta, r\sin\theta, 0)$ be a chart for the upper half plane S_1. Let $f : \mathbb{R}_{>0} \times (0, \pi) \to S_2$ be defined by

$$f(r,\theta) = \left(\frac{r}{\sqrt{2}}\cos\sqrt{2}\theta, \frac{r}{\sqrt{2}}\sin\sqrt{2}\theta, \frac{r}{\sqrt{2}}\right).$$

Then $g = f \circ X^{-1}$ is a well-defined smooth map from S_1 to S_2. It remains to check that g is a local isometry. We have

$$\begin{aligned}
X_r &= (\cos\theta, \sin\theta, 0) \\
X_\theta &= (-r\sin\theta, r\cos\theta, 0) \\
\langle X_r, X_r \rangle &= 1 \\
\langle X_r, X_\theta \rangle &= 0 \\
\langle X_\theta, X_\theta \rangle &= r^2
\end{aligned}$$

while

$$\begin{aligned}
dg(X_r) &= \frac{\partial f}{\partial r} = \left(\frac{1}{\sqrt{2}}\cos\sqrt{2}\theta, \frac{1}{\sqrt{2}}\sin\sqrt{2}\theta, \frac{1}{\sqrt{2}}\right) \\
dg(X_\theta) &= \frac{\partial f}{\partial \theta} = \left(-r\sin\sqrt{2}\theta, r\cos\sqrt{2}\theta, 0\right) \\
\langle dg(X_r), dg(X_r) \rangle &= 1 \\
\langle dg(X_r), dg(X_\theta) \rangle &= 0 \\
\langle dg(X_\theta), dg(X_\theta) \rangle &= r^2.
\end{aligned}$$

It follows that g is indeed a local isometry.

To find the mean and Gauss curvatures of S_2. We can use the standard chart

$$Y : (r, \theta) \mapsto (r\cos\theta, r\sin\theta, r)$$

or f defined above or many others. Let us use the first one. We have

$$\begin{aligned}
Y_r &= (\cos\theta, \sin\theta, 1) \\
Y_\theta &= (-r\sin\theta, r\cos\theta, 0) \\
Y_r \times Y_\theta &= (-r\cos\theta, -r\sin\theta, r) \\
N &= \frac{1}{\sqrt{2}}(-\cos\theta, -\sin\theta, 1) \quad \text{(upward)}
\end{aligned}$$

$$\begin{aligned}
\mathcal{S}(Y_r) &= 0 \cdot Y_r + 0 \cdot Y_\theta \\
\mathcal{S}(Y_\theta) &= \frac{1}{\sqrt{2}}(-\sin\theta, \cos\theta, 0) = 0 \cdot Y_r + \frac{1}{\sqrt{2}r} Y_\theta
\end{aligned}$$

$$[\mathcal{S}]_{\{Y_r, Y_\theta\}} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{\sqrt{2}r} \end{pmatrix}$$

$$H = \frac{1}{\sqrt{2}r} \qquad K = 0$$

Remark. The above map is just the mathematical description of a childhood craft, namely making a cone from a sheet of circular sector. Recall that we form the cone simply by gluing the two boundary half-lines of the sector together. In our case, in order to form S_2, we in fact need a sector with larger angle so that it contains the ray $\theta = \sqrt{2}\pi$ which is to be glued to the positive x-axis.

7. (2 points) Recall from Q2 above that the helicoid S_1 and the catenoid S_2 are given by

$$\begin{aligned} X(u,v) &= (v\cos u, v\sin u, u), && (u,v) \in \mathbb{R}^2, \\ Y(u,v) &= (\cosh v \cos u, \cosh v \sin u, v), && (u,v) \in (0, 2\pi) \times \mathbb{R} \end{aligned}$$

respectively. Note that X is a diffeomorphism onto the whole surface S_1 (while Y misses a catenary curve). To define a local ismoetry from the helicoid to the catenoid, it suffices to find a map $\phi : \mathbb{R}^2 \to S_2$ and show that the "pull-backs" of the first fundamental forms of S_1 via X and S_2 via ϕ are the same. We have already found the former: $g = \begin{pmatrix} 1+v^2 & 0 \\ 0 & 1 \end{pmatrix}$. For the latter, we let

$$\phi : \mathbb{R}^2 \to S_2 : (u,v) \mapsto (\sqrt{1+v^2}\cos u, \sqrt{1+v^2}\sin u, \sinh^{-1} v).$$

Let $\psi = \phi \circ X^{-1} : S_1 \to S_2$. We have

$$\begin{aligned} d\psi(X_u) &= \frac{\partial \phi}{\partial u} = (-\sqrt{1+v^2}\sin u, \sqrt{1+v^2}\cos u, 0) \\ d\psi(X_v) &= \frac{\partial \phi}{\partial v} = \left(\frac{v}{\sqrt{1+v^2}}\cos u, \frac{v}{\sqrt{1+v^2}}\sin u, \frac{1}{\sqrt{1+v^2}} \right) \\ \langle d\psi(X_u), d\psi(X_u) \rangle &= 1+v^2 \\ \langle d\psi(X_u), d\psi(X_v) \rangle &= 0 \\ \langle d\psi(X_v), d\psi(X_v) \rangle &= 1. \end{aligned}$$

It follows that ψ is a local isometry from the helicoid to the catenoid.

However, they are not globally isometric because the helicoid is contractible (as it is homeomorphic to $\mathbb{R}^2$) while the catenoid is homotopy equivalent to the circle which has non-zero fundamental group. There is another method which is provided by two of you: From the calculations in Q2, we see that the sets of points p in the helicoid and in the catenoid such that $K(p) = -1$ are $\{(0,0,z) | \ z \in \mathbb{R}\}$ and $\{(x,y,0) | \ x^2 + y^2 = 1\}$ respectively. If the two surfaces are globally isometric, then these two sets are homeomorphic. However, this is not true because one is an infinite straight line which is non-compact while the other one is a circle which is compact.

Remark 1. The map $\psi : S_1 \to S_2$ is the universal covering with deck transformations freely generated by the translation by 2π units upward.

Remark 2. Concerning the last part of this question which is about whether the two surfaces are globally isometric or not, I could only think of the first method which is beyond the scope of this course, so originally I intended not to grade this part. But thanks to our two classmates who provided such a nice solution, I finally decided to count this part which is now worth 0.5 points.