MATH 4030 Differential Geometry
Homework 3
Suggested solutions

1. (1 point)
Solution 1. Let $S' = \{(f(u)\cos v, f(u)\sin v, g(u))|\ u \in \mathbb{R}, v \in [-\pi,\pi)\}$. Then S is open in S'. We show that S' is the zero set of a smooth function of which 0 is a regular value, and hence S', as well as S, is a surface. To construct the function, observe that $g' > 0$ implies that g is a diffeomorphism onto an open interval I. Let $F : \{(x,y,z)|\ x^2+y^2 > 0, z \in I\} \to \mathbb{R}$ be defined by

$$F(x,y,z) = \sqrt{x^2+y^2} - f(g^{-1}(z)).$$

Then

$$\nabla F(x,y,z) = \left[\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, -\frac{f'(g^{-1}(z))}{g'(g^{-1}(z))}\right],$$

and hence every real number is a regular value of F (we call such function a *submersion*). This shows that S is a surface.

Now recall that ∇F is normal to S. Observe also that the (x,y)-components of ∇F are multiple of (x,y), and hence that the line through (x,y,z) parallel to ∇F passes through the z-axis.

Solution 2. Consider the function X, we show that it is a parametrization. We have

$$\begin{aligned} X_u &= (f'(u)\cos v, f'(u)\sin v, g'(u)) \\ X_v &= (-f(u)\sin v, f(u)\cos v, 0) \\ X_u \times X_v &= (-f(u)g'(u)\cos v, -f(u)g'(u)\sin v, f(u)f'(u)) \end{aligned}$$

Since $f, g' > 0$, we have $|X_u \times X_v| = f\sqrt{f'^2 + g'^2} \neq 0$, and hence dX is one-to-one. Now comes the injectivity: Suppose $X(u_1,v_1) = X(u_2,v_2)$, then

$$\begin{aligned} g(u_1) &= g(u_2) \\ f(u_1)e^{iv_1} &= f(u_2)e^{iv_2} \end{aligned}$$

Since $g'(u) > 0\ \ \forall u \in \mathbb{R}$, we have $u_1 = u_2$, and so

$$\begin{aligned} e^{iv_1} &= e^{iv_2} \\ v_1 &= v_2. \end{aligned}$$

As for showing that X has continuous inverse (homeomorphism), we cheat by using the fact that if S has been proved to be a surface (see solution 1), then any X such that (1) X is injective and (2) dX is one-to-one everywhere is a chart. This basically follows from the inverse function theorem. Hence X is a chart of S.

Now our chart X allows us to show the last thing: all the normal lines of S pass through the z-axis. To do this, recall that

$$w := X_u \times X_v = (-f(u)g'(u)\cos v, -f(u)g'(u)\sin v, f(u)f'(u))$$

is normal to S. Then observe that

$$X(u,v) + \frac{1}{g'(u)}w = \left(0, 0, g(u) + \frac{f(u)f'(u)}{g'(u)}\right).$$

This means that the normal line of S at $X(u,v)$ pass through a point in the z-axis, as was to be shown.

2. (1 point)

   (a) Let $X : U \to V \subseteq S$ be any chart of S. Consider the function

$$f \circ X : U \to \mathbb{R} : (u,v) \mapsto \sqrt{\langle X(u,v) - p_0, X(u,v) - p_0 \rangle}.$$

   Then $f \circ X$ is smooth if and only if $\langle X(u,v) - p_0, X(u,v) - p_0 \rangle > 0$ for any $(u,v) \in U$. The latter is true because of the assumption $p_o \notin S$, and so we get the first half of (a).

   Let $\zeta \in T_pS$. Choose $\alpha : (-\varepsilon, \varepsilon) \to S$ such that $\alpha(0) = p$ and $\alpha'(0) = \zeta$. Then

$$\begin{aligned} (f \circ \alpha)(t) &= \sqrt{\langle \alpha(t) - p_0, \alpha(t) - p_0 \rangle} \\ df_p(\zeta) = (f \circ \alpha)'(0) &= \frac{1}{|\alpha(0) - p_0|} \langle \alpha'(0), \alpha(0) - p_0 \rangle \\ &= \frac{1}{|p - p_0|} \langle \zeta, p - p_0 \rangle. \end{aligned}$$

   It follows that $df_p \equiv 0$ if and only if $\langle \zeta, p - p_0 \rangle = 0$ for any $\zeta \in T_pS$ which is equivalent to $p - p_0$ being normal to S at p.

   (b) To avoid confusion, I change v in question to θ. Let $X : U \to V \subseteq S$ be any chart of S. Then it is clear that the function

$$h \circ X : U \to \mathbb{R} : (u,v) \mapsto \langle X(u,v), \theta \rangle$$

   is smooth on U. This proves the first half of (b).

   Let $\zeta \in T_pS$. Choose $\alpha : (-\varepsilon, \varepsilon) \to S$ such that $\alpha(0) = p$ and $\alpha'(0) = \zeta$. Then

$$\begin{aligned} (h \circ \alpha)(t) &= \langle \alpha(t), \theta \rangle \\ dh_p(\zeta) &= (h \circ \alpha)'(0) \\ &= \langle \alpha'(0), \theta \rangle \\ &= \langle \zeta, \theta \rangle. \end{aligned}$$

   It follows that $dh_p \equiv 0$ if and only if $\langle \zeta, \theta \rangle = 0$ for any $\zeta \in T_pS$ which is equivalent to θ being normal to S at p.

3. (1.5 points)

   (a) The idea is to first prepare a plane normal to a that is far away from S. Then translate this plane towards S until it touches S for the first time. The point of tangency will be the point we are looking for.

   To put this idea into mathematical language, we "parametrize" all planes normal to a by the linear map $h : p \mapsto \langle p, a \rangle$. The level sets of this map are exactly the set of all planes normal to a. The remote plane that we prepared in the first place is $h^{-1}(c_0)$ for any large $c_0 > 0$. Translating this plane towards S means we consider the family

$h^{-1}(c)$ of planes as c decreases starting from c_0. The first moment when the plane touches S is the moment when $h^{-1}(c)$ first intersects S which is the same as when c first enters $h(S)$.

Now it is clear that this happens exactly when $h|_S$ attains maximum, which is guaranteed by the compactness of S. Let p_0 be a point at which h attains maximum. Then p_0 is a critical point of $h|_S$, and hence by Q2b above, we conclude that a is normal to S at p_0. In other words, the normal line to S at p_0 is parallel to a, as desired.

(b) Suppose all the normal lines of S are parallel to a. Then the function $h|_S$ with h defined in part (a) has zero derivative everywhere. Since S is connected, it follows that $h|_S$ is a constant function, in other words, S is contained in some level set of h which is a plane.

Remark 1. Note that we don't need the compactness assumption for part (b). If S is compact, then it must be a surface-with-boundary such as disk and annulus.

Remark 2. In part (b), or elsewhere, some of you have used the fact that if S is connected (i.e. S cannot be covered by two disjoint non-empty open subsets), then every pair of points in S can be joined by a SMOOTH curve (in fact, by a smooth arc). It should be noted that this fact is non-trivial. However, since its proof does not involve anything important to this course, you are welcome to use it (without giving a proof) whenever you like.

4. (1 point)

(a) Apply HW3 suggested ex. Q3 (whose solution can be found in Tut5) to f and to f^{-1}. (A diffeomorphism is a local diffeomorphism!)

(b) First we recall the following two equivalent definitions of an orientation-preserving (resp. orientation-reversing) diffeomorphism $f : S_1 \to S_2$:

(A) For each $p \in S_1$, there exists a pair of linearly independent tangent vectors $\zeta, \theta \in T_pS_1$ such that $\mathcal{G}_1(p) = \dfrac{\zeta \times \theta}{|\zeta \times \theta|}$ and $\mathcal{G}_2(f(p)) = (\text{resp.}-)\dfrac{df_p(\zeta) \times df_p(\theta)}{|df_p(\zeta) \times df_p(\theta)|}$. ($\mathcal{G}_1$ and $\mathcal{G}_2$ are fixed Gauss maps for S_1 and S_2 respectively.)

(B) For each pair of charts $X_i : U_i \to V_i \subseteq S_i$, $i = 1, 2$ belonging to an atlas Φ_i determining the given orientation on S_i, $X_2^{-1} \circ f \circ X_1$ has positive (resp. negative) Jacobian at every point $(u, v) \in U_1$ at which this map is well-defined (i.e. $f(X_1(u, v)) \in V_2$).

Recall also that the correspondence between $\mathcal{G}$ and Φ is as follows:

- Given $\mathcal{G}$, we take Φ to be the collection of all charts $X : U \to V \subseteq S$ such that $\mathcal{G} = \dfrac{X_u \times X_v}{|X_u \times X_v|}$.
- Given Φ, then the unit normal vector field $\dfrac{X_u \times X_v}{|X_u \times X_v|}$ is consistent among different choices of charts $X \in \Phi$. Take $\mathcal{G}$ to be the well-defined map glued by these vector fields.

(To see why the two definitions are equivalent and why the correspondence makes sense, we may need the results in Q5 below and Tut5.)

Solution 1. (More comprehensible but less detailed) Let $p \in \mathbb{S}^2$ and N be the outward normal vector field on $\mathbb{S}^2$. Take orthonormal basis ζ, θ of $T_p\mathbb{S}^2$ such that $N_p = \zeta \times \theta$. Then $N_{f(p)} = N_{-p} = -N_p = -\zeta \times \theta$ (note that as subspaces of $\mathbb{R}^3$, $T_p\mathbb{S}^2 = T_{f(p)}\mathbb{S}^2$). But $df_p(\zeta) = -\zeta$ and $df_p(\theta) = -\theta$ so that $df_p(\zeta) \times df_p(\theta) = \zeta \times \theta = -N_{f(p)}$. (A) implies f is an orientation-reversing diffeomorphism.

Solution 1'. (Less comprehensible but more detailed) Now, let f be the antipodal map on $S_1 = S_2 = \mathbb{S}^2$. Recall (say from Tut4) that the unit sphere $\mathbb{S}^2$ can be covered by 6 graphical charts (2 for each coordinate plane). We consider one of them $X : \{u^2 + v^2 < 1\} \to \mathbb{S}^2 \subseteq \mathbb{R}^3 : (u, v) \mapsto (u, v, \sqrt{1 - u^2 - v^2})$. Then

$$df(X_u) = \left(-1, 0, \frac{u}{\sqrt{1 - u^2 - v^2}}\right)$$
$$df(X_v) = \left(0, -1, \frac{v}{\sqrt{1 - u^2 - v^2}}\right).$$

Note that $\dfrac{X_u \times X_v}{|X_u \times X_v|} = \mathcal{G}$ (where we take our Gauss map $\mathcal{G}$ to be the outward normal $\mathbb{S}^2 \ni (x, y, z) \mapsto (x, y, z)$), but

$$df(X_u) \times df(X_v) = \left(\frac{u}{\sqrt{1 - u^2 - v^2}}, \frac{v}{\sqrt{1 - u^2 - v^2}}, 1\right)$$

so that

$$\frac{df(X_u) \times df(X_v)}{|df(X_u) \times df(X_v)|} = (u, v, \sqrt{1 - u^2 - v^2}) = -\mathcal{G}_{f(X(u,v))}$$

(note that $f(X(u, v)) = (-u, -v, -\sqrt{1 - u^2 - v^2})$). It follows that (A) for orientation-reversing diffeomorphism is satisfied over the region $\mathbb{S}^2 \cap \{z > 0\}$. Similarly, (A) is satisfied over the other five regions similarly defined which cover the whole $\mathbb{S}^2$ altogether.

5. (1 point) Since $X = \overline{X} \circ (\overline{X}^{-1} \circ X) = \overline{X} \circ \psi$, by chain rule, we have $dX = d\overline{X} \circ d\psi$. Plugging the column vector $\mathbf{a} = (a_1, a_2)^T$ into both sides, we get

$$a_1 \frac{\partial X}{\partial u} + a_2 \frac{\partial X}{\partial v} = dX(\mathbf{a}) = d\overline{X}(d\psi(\mathbf{a})) = d\overline{X}\left[\begin{pmatrix} \frac{\partial \bar{u}}{\partial u} & \frac{\partial \bar{u}}{\partial v} \\ \frac{\partial \bar{v}}{\partial u} & \frac{\partial \bar{v}}{\partial v} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}\right].$$

The LHS is also equal to $b_1 \dfrac{\partial \overline{X}}{\partial \bar{u}} + b_2 \dfrac{\partial \overline{X}}{\partial \bar{v}}$ which is equal to $d\overline{X}\left[\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}\right]$. By the injectivity of $d\overline{X}$, we have

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} \frac{\partial \bar{u}}{\partial u} & \frac{\partial \bar{u}}{\partial v} \\ \frac{\partial \bar{v}}{\partial u} & \frac{\partial \bar{v}}{\partial v} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}.$$

6. (1 point) Let $X : (0, 2\pi) \times (0, 2\pi) \to S - \infty$:

$$(\theta, \varphi) \mapsto ((a + r\cos\theta)\cos\varphi, (a + r\cos\theta)\sin\varphi, r\sin\theta)$$

be a parametrization of S where ∞ denotes the union of two circles which has measure 0. Then

$$\begin{aligned}
X_\theta &= (-r\sin\theta\cos\varphi, -r\sin\theta\sin\varphi, r\cos\theta) \\
X_\varphi &= (-(a + r\cos\theta)\sin\varphi, (a + r\cos\theta)\cos\varphi, 0) \\
g_{\theta\theta} &= \langle X_\theta, X_\theta \rangle = r^2 \\
g_{\theta\varphi} &= \langle X_\theta, X_\varphi \rangle = 0 \\
g_{\varphi\varphi} &= \langle X_\varphi, X_\varphi \rangle = (a + r\cos\theta)^2
\end{aligned}$$

The area of S is equal to

$$\begin{aligned}
&\int_0^{2\pi}\int_0^{2\pi} \sqrt{\det(g_{ij})}\ d\theta d\varphi \\
&= \int_0^{2\pi}\int_0^{2\pi} r(a + r\cos\theta)\ d\theta d\varphi \\
&= 4\pi^2 ar.
\end{aligned}$$

7. (1.5 points) Consider the chart $X : \mathbb{R}^2 \to \mathbb{R}^3 : (u, v) \mapsto (u, v, f(u, v))$ where f is smooth. Then

$$\begin{aligned}
X_u &= (1, 0, f_u) \\
X_v &= (0, 1, f_v) \\
X_{uu} &= (0, 0, f_{uu}) \\
X_{vv} &= (0, 0, f_{vv}) \\
X_{uv} &= (0, 0, f_{uv}) \\
X_u \times X_v &= (-f_u, -f_v, 1)
\end{aligned}$$

$$N = \frac{X_u \times X_v}{|X_u \times X_v|} = \frac{(-f_u, -f_v, 1)}{\sqrt{f_u^2 + f_v^2 + 1}} \text{ (upward normal)}$$

$$g = \begin{pmatrix} \langle X_u, X_u \rangle & \langle X_u, X_v \rangle \\ \langle X_v, X_u \rangle & \langle X_v, X_v \rangle \end{pmatrix} = \begin{pmatrix} 1 + f_u^2 & f_u f_v \\ f_u f_v & 1 + f_v^2 \end{pmatrix}$$

$$A = \begin{pmatrix} \langle N, X_{uu} \rangle & \langle N, X_{uv} \rangle \\ \langle N, X_{vu} \rangle & \langle N, X_{vv} \rangle \end{pmatrix} = \begin{pmatrix} \frac{f_{uu}}{\sqrt{f_u^2+f_v^2+1}} & \frac{f_{uv}}{\sqrt{f_u^2+f_v^2+1}} \\ \frac{f_{uv}}{\sqrt{f_u^2+f_v^2+1}} & \frac{f_{vv}}{\sqrt{f_u^2+f_v^2+1}} \end{pmatrix}$$

$$[\mathcal{S}]_{\{X_u, X_v\}} = g^{-1}A = \frac{1}{\sqrt{f_u^2 + f_v^2 + 1}^3} \begin{pmatrix} (1 + f_v^2)f_{uu} - f_u f_v f_{uv} & (1 + f_v^2)f_{uv} - f_u f_v f_{vv} \\ (1 + f_u^2)f_{uv} - f_u f_v f_{uu} & (1 + f_u^2)f_{vv} - f_u f_v f_{uv} \end{pmatrix}$$

($[\mathcal{S}]_{\{X_u,X_v\}}$ can also be found by expressing $-\partial_u N$ and $-\partial_v N$ in terms of X_u and X_v.)

$$\begin{aligned} H &= tr(g^{-1}A) \\ &= \frac{(1+f_v^2)f_{uu} - 2f_u f_v f_{uv} + (1+f_u^2)f_{vv}}{(f_u^2+f_v^2+1)^{\frac{3}{2}}} \end{aligned}$$

$$\begin{aligned} K &= \frac{\det A}{\det g} \\ &= \frac{\frac{f_{uu}f_{vv}-f_{uv}^2}{(f_u^2+f_v^2+1)}}{(f_u^2+f_v^2+1)} \\ &= \frac{f_{uu}f_{vv}-f_{uv}^2}{(f_u^2+f_v^2+1)^2} \end{aligned}$$

$S_1 = \{(x,y,z) \in \mathbb{R}^3 | \ z = x^2+y^2\}$:

- Let $X : \mathbb{R}^2 \to S_1 : (u,v) \mapsto (u,v,u^2+v^2)$ be a parametrization of S_1. Let $f(u,v) = u^2+v^2$. We have

$$\begin{aligned} f_u = 2u, \quad f_v = 2v \\ f_{uu} = 2, \quad f_{vv} = 2, \quad f_{uv} = 0 \end{aligned}$$

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$$\begin{aligned} H &= \frac{(1+f_v^2)f_{uu} - 2f_u f_v f_{uv} + (1+f_u^2)f_{vv}}{(f_u^2+f_v^2+1)^{\frac{3}{2}}} \\ &= \frac{2(4u^2+4v^2+2)}{(4u^2+4v^2+1)^{\frac{3}{2}}} \\ &= \frac{4(2u^2+2v^2+1)}{(4u^2+4v^2+1)^{\frac{3}{2}}} \end{aligned}$$

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$$\begin{aligned} K &= \frac{f_{uu}f_{vv}-f_{uv}^2}{(f_u^2+f_v^2+1)^2} \\ &= \frac{4}{(4u^2+4v^2+1)^2} \end{aligned}$$

- The second fundamental form A at $p = (0,0,0)$ is $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$.
- The matrix representing $\mathcal{S}_p$ with respect to the ordered basis $\{X_u, X_v\}$ is

$$g^{-1}A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

- Principal curvatures = eigenvalues of S_p:

$$\kappa_1 = \kappa_2 = 2$$

- Principal directions = (normalized) eigenvectors of S_p $= \{(a,b,0) \in \mathbb{R}^3 |\ a^2+b^2=1\}$.

$S_2 = \{(x,y,z) \in \mathbb{R}^3 |\ z = x^2 - y^2\}$:

- Let $X : \mathbb{R}^2 \to S_1 : (u,v) \mapsto (u,v,u^2-v^2)$ be a parametrization of S_1. Let $f(u,v) = u^2 - v^2$. We have

$$\begin{aligned} f_u = 2u, \quad f_v = -2v \\ f_{uu} = 2, \quad f_{vv} = -2, \quad f_{uv} = 0 \end{aligned}$$

- $$\begin{aligned} H &= \frac{(1+f_v^2)f_{uu} - 2f_uf_vf_{uv} + (1+f_u^2)f_{vv}}{(f_u^2+f_v^2+1)^{\frac{3}{2}}} \\ &= \frac{8(v^2-u^2)}{(4u^2+4v^2+1)^{\frac{3}{2}}} \end{aligned}$$

- $$\begin{aligned} K &= \frac{f_{uu}f_{vv} - f_{uv}^2}{(f_u^2+f_v^2+1)^2} \\ &= -\frac{4}{(4u^2+4v^2+1)^2} \end{aligned}$$

- The second fundamental form A at $p = (0,0,0)$ is $\begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$.
- The matrix representing $\mathcal{S}_p$ with respect to the ordered basis $\{X_u, X_v\}$ is
$$g^{-1}A = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}.$$
- Principal curvatures = eigenvalues of S_p:
$$\kappa_1 = 2,\ \kappa_2 = -2$$
- Principal directions = (normalized) eigenvectors of S_p:
$$\nu_1 = \pm\frac{1 \cdot X_u + 0 \cdot X_v}{|1 \cdot X_u + 0 \cdot X_v|} = (\pm 1, 0, 0)$$
$$\nu_2 = \pm\frac{0 \cdot X_u + 1 \cdot X_v}{|0 \cdot X_u + 1 \cdot X_v|} = (0, \pm 1, 0)$$

8. (1 point) Observe that any symmetric 2×2 real matrix $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ with zero trace and zero determinant must be zero:
$$\begin{cases} a + c &= 0 \\ ac - b^2 &= 0 \end{cases} \Longrightarrow \begin{cases} a + c &= 0 \\ a^2 + b^2 &= 0 \end{cases} \Longrightarrow a = b = c = 0.$$

The same is thus true for linear operators on $\mathbb{R}^2$ symmetric with respect to an inner product.

Now apply this to the shape operator $\mathcal{S} = -dN_p : T_pS \to T_pS$ for each point p in our surface S. Recall that H and K are the trace and the determinant of $\mathcal{S}$ respectively, and that $\mathcal{S}$ is symmetric with respect to the inner product induced from the standard one on $\mathbb{R}^3$. That H and S are both identically zero implies that $\mathcal{S}$ is identically zero, by the above fact. In other words, the Gauss map N is constant (assuming S is connected), and hence the assumption in Q3(b) above is satisfied, proving that S is contained in a plane.

9. (1 point) First note that ellipsoid, similar to sphere, can be covered by six graphical charts. Recall from the solution to Q7 that for the graphical surface defined by a function f, we have the following formula for its Gauss curvature:

$$K = \frac{f_{uu}f_{vv} - f_{uv}^2}{(f_u^2 + f_v^2 + 1)^2}.$$

Therefore, it suffices to show that $f_{uu}f_{vv} - f_{uv}^2 > 0$ for

$$f : \left\{ \frac{u^2}{a^2} + \frac{v^2}{b^2} < 1 \right\} \to \mathbb{R} : (u, v) \mapsto c\sqrt{1 - \frac{u^2}{a^2} - \frac{v^2}{b^2}}.$$

(Other five are handled similarly.) This can be checked by direct calculation which I leave to you. Alternatively, write f as $c \cdot g \circ A$ where $g : (x, y) \mapsto \sqrt{1 - x^2 - y^2}$ and $A : (u, v) \mapsto (u/a, v/b)$. Then the Hessians of f and g are related by the following equality:

$$Hess(f)_{(u,v)} = cA^T \circ Hess(g)_{(Au,Av)} \circ A,$$

where the linear map A is regarded as a 2×2 matrix. Now it is clear that

$$f_{uu}f_{vv} - f_{uv}^2 = \det(Hess(f)) = \frac{c^2}{a^2b^2} \det(Hess(g)).$$

Of course, we can check by computation that $\det(Hess(g)) > 0$ so that the result follows, but this is unnecessary because this expression corresponds to the numerator of the Gauss curvature of the unit sphere which we know is positive everywhere. The desired result thus follows from this observation.