MATH 4030 Differential Geometry
Homework 2
Suggested solutions

1. (1 point)

   (a) $\alpha'(\theta) = (r'(\theta)\cos\theta - r(\theta)\sin\theta, r'(\theta)\sin\theta + r(\theta)\cos\theta)$

   $|\alpha'(\theta)| = \sqrt{(r'(\theta)\cos\theta - r(\theta)\sin\theta)^2 + (r'(\theta)\sin\theta + r(\theta)\cos\theta)^2} = \sqrt{r(\theta)^2 + r'(\theta)^2}$

   $L_a^b(\alpha) = \int_a^b \sqrt{r(\theta)^2 + r'(\theta)^2}\, d\theta$

   (b) By HW1 Q6, we have
$$\kappa(\theta) = \frac{\det\big(\alpha'(\theta), \alpha''(\theta)\big)}{|\alpha'(\theta)|^3}.$$

   $\alpha''(\theta) = (r''(\theta)\cos\theta - 2r'(\theta)\sin\theta - r(\theta)\cos\theta, r''(\theta)\sin\theta + 2r'(\theta)\cos\theta - r(\theta)\sin\theta)$

   $\det\big(\alpha'(\theta), \alpha''(\theta)\big) = \det\begin{pmatrix} r'c - rs & r''c - 2r's - rc \\ r's + rc & r''s + 2r'c - rs \end{pmatrix} = 2r'(\theta)^2 - r(\theta)r''(\theta) + r(\theta)^2$

   Hence
$$\kappa(\theta) = \frac{2r'(\theta)^2 - r(\theta)r''(\theta) + r(\theta)^2}{[r'(\theta)^2 + r(\theta)^2]^{3/2}}.$$

2. (2 points)

   (a) $(\Longrightarrow)$
   Suppose α lies on a plane. Let p be a point on the plane and v be a unit vector normal to the plane. Then
$$\begin{aligned} \langle \alpha(s) - p, v\rangle &\equiv 0 \\ \langle T(s), v\rangle = \langle \alpha'(s), v\rangle &\equiv 0 \\ \kappa(s)\langle N(s), v\rangle = \langle \alpha''(s), v\rangle &\equiv 0. \end{aligned}$$

   Since $\kappa(s) > 0$ for all $s \in I$, we have $\langle N(s), v\rangle = 0$ for all $s \in I$. It follows that $B(s) = \pm v$ for all $s \in I$, i.e. $B(s)$ is a constant vector. Therefore
$$\tau(s) = \langle B'(s), N(s)\rangle = 0 \ \text{ for all } s \in I.$$

   $(\Longleftarrow)$
   If $\tau(s) \equiv 0$, then $B'(s) = 0$ for all $s \in I$, and hence $B(s)$ is a constant vector. Let $v = B(s)$. Consider $\langle \alpha(s) - \alpha(0), v\rangle$ (just assume $0 \in I$). We have
$$\langle \alpha(s) - \alpha(0), v\rangle' = \langle \alpha'(s), v\rangle = \langle T(s), B(s)\rangle = 0 \ \text{ for all } s \in I.$$

   Hence $\langle \alpha(s) - \alpha(0), v\rangle = c$ for some constant c. Put $s = 0$, we get $c = \langle \alpha(0) - \alpha(0), v\rangle = 0$. Therefore, $\alpha(s)$ is contained in the plane passing through $\alpha(0)$ normal to v.

(b) ($\Longrightarrow$)

If α is an arc of a circular helix or a circle, then by a rigid motion, we can assume that

$$\alpha(s) = \left(r\cos\left(\frac{s}{\sqrt{r^2+c^2}}\right), r\sin\left(\frac{s}{\sqrt{r^2+c^2}}\right), \frac{cs}{\sqrt{r^2+c^2}}\right)$$

for some constants r and c. Note that α is p.b.a.l.

Then

$$\begin{aligned}
\alpha'(s) &= \left(-\frac{r}{\sqrt{r^2+c^2}}\sin\left(\frac{s}{\sqrt{r^2+c^2}}\right), \frac{r}{\sqrt{r^2+c^2}}\cos\left(\frac{s}{\sqrt{r^2+c^2}}\right), \frac{c}{\sqrt{r^2+c^2}}\right) \\
\alpha''(s) &= \left(-\frac{r}{r^2+c^2}\cos\left(\frac{s}{\sqrt{r^2+c^2}}\right), -\frac{r}{r^2+c^2}\sin\left(\frac{s}{\sqrt{r^2+c^2}}\right), 0\right) \\
\kappa(s) &= |\alpha''(s)| = \frac{r}{r^2+c^2} \\
N(s) &= \left(-\cos\left(\frac{s}{\sqrt{r^2+c^2}}\right), -\sin\left(\frac{s}{\sqrt{r^2+c^2}}\right), 0\right) \\
B(s) &= \left(\frac{c}{\sqrt{r^2+c^2}}\sin\left(\frac{s}{\sqrt{r^2+c^2}}\right), -\frac{c}{\sqrt{r^2+c^2}}\cos\left(\frac{s}{\sqrt{r^2+c^2}}\right), \frac{r}{\sqrt{r^2+c^2}}\right) \\
B'(s) &= \left(\frac{c}{r^2+c^2}\cos\left(\frac{s}{\sqrt{r^2+c^2}}\right), \frac{c}{r^2+c^2}\sin\left(\frac{s}{\sqrt{r^2+c^2}}\right), 0\right) \\
\tau(s) &= -\frac{c}{r^2+c^2}
\end{aligned}$$

It follows that κ and τ are constant.

($\Longleftarrow$)

Suppose κ and τ are constant. Let $r = \frac{\kappa}{\kappa^2+\tau^2}$ and $c = -\frac{\tau}{\kappa^2+\tau^2}$.

Consider

$$\beta(s) = \left(r\cos\left(\frac{s}{\sqrt{r^2+c^2}}\right), r\sin\left(\frac{s}{\sqrt{r^2+c^2}}\right), \frac{cs}{\sqrt{r^2+c^2}}\right)$$

By the above calculations, we have $\kappa_\beta = \kappa$ and $\tau_\beta = \tau$. Hence, by the Fundamental Theorem of Space Curves, $\beta = \varphi \circ \alpha$ for some orientation-preserving rigid motion φ. It follows that α is an arc of a circular helix or a circle.

3. (1 point)

$$\alpha'(t) = |\alpha'(t)|T(t)$$

$$\begin{aligned}\alpha''(t) &= \frac{d|\alpha'(t)|}{dt}T(t) + |\alpha'(t)|\frac{ds}{dt}\frac{dT}{ds} \\ &= \frac{d|\alpha'(t)|}{dt}T(t) + |\alpha'(t)|^2\kappa(t)N(t)\end{aligned}$$

$$\begin{aligned}\alpha'(t)\times\alpha''(t) &= |\alpha'(t)|^2\kappa(t)\alpha'(t)\times N(t) \\ &= |\alpha'(t)|^3\kappa(t)B(t)\end{aligned}$$

$$\kappa(t) = \frac{|\alpha'(t)\times\alpha''(t)|}{|\alpha'(t)|^3}$$

$$B(t) = \frac{\alpha'(t)\times\alpha''(t)}{|\alpha'(t)\times\alpha''(t)|}$$

$$\begin{aligned}\alpha'''(t) = {}& \frac{d^2|\alpha'(t)|}{dt^2}T(t) + \frac{d|\alpha'(t)|}{dt}|\alpha'(t)|\kappa(t)N(t) \\ &+ \frac{d(|\alpha'(t)|^2\kappa(t))}{dt}N(t) \\ &+ |\alpha'(t)|^3\kappa(t)\big(-\kappa(t)T(t) - \tau(t)B(t)\big)\end{aligned}$$

$$\langle B(t), \alpha'''(t)\rangle = |\alpha'(t)|^3\kappa(t)(-\tau(t))$$

$$\begin{aligned}\tau(t) &= -\frac{\langle B(t), \alpha'''(t)\rangle}{\kappa(t)|\alpha'(t)|^3} \\ &= -\frac{\langle \alpha'(t)\times\alpha''(t), \alpha'''(t)\rangle}{|\alpha'(t)\times\alpha''(t)|^2}\end{aligned}$$

4. (2 points) Recall that for a curve $\alpha : I \to \mathbb{R}^3$ p.b.a.l. with $\kappa(s) > 0$ for all $s \in I$, we have

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & -\tau \\ 0 & \tau & 0 \end{bmatrix}\begin{bmatrix} T \\ N \\ B \end{bmatrix}.$$

If $\tau(s) \equiv c\kappa(s)$ for a constant c, then the matrix in the above equation is a (possibly non-constant) scalar multiple of a constant matrix, namely we can write it as $\kappa(s)A$ where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & -c \\ 0 & c & 0 \end{bmatrix}.$$

Observe that A is singular for any values of c because $[c\ 0\ -1]A = 0$. Therefore, if we let $u(s) = cT(s) - B(s)$, then

$$\begin{aligned} u'(s) &= \begin{bmatrix} c & 0 & -1 \end{bmatrix} \begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} \\ &= \begin{bmatrix} c & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & -\tau \\ 0 & \tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix} \\ &= \kappa(s)[c\ 0\ -1]A[T\ N\ B]^T = 0, \end{aligned}$$

and hence u is constant. This proves ($\Longleftarrow$) after normalizing u to get v.

For ($\Longrightarrow$), assume $\langle \alpha'(s), v\rangle \equiv c'$ where c' is constant. We see in the last part that the vector v, when expressed with respect to the frame $\{T(s), N(s), B(s)\}$, should have constant coefficients. Indeed, this is the case because $\langle v, T\rangle = \langle v, \alpha'\rangle = c'$, $\langle v, N\rangle = \frac{1}{\kappa}\langle v, \alpha'\rangle' = 0$ and $\langle v, B\rangle' = \tau\langle v, N\rangle = 0$. Let c'' be the last constant, $\langle v, B\rangle$. Then $v = c'T + c''B$. Note that $|v| = 1$ implies that c' and c'' cannot be both zero. Now differentiating the last equality, we get $0 = c'T' + c''B' = (c'\kappa + c''\tau)N$, and hence

$$c'\kappa + c''\tau \equiv 0.$$

Note that c'' cannot be zero, for otherwise c' will be zero ($\because\ \kappa > 0$). It follows that $\tau \equiv (-\frac{c'}{c''})\kappa$. (Take $c = -\frac{c'}{c''}$.)

5. (1 point)
Solution 1. The point which causes problem is the origin $\mathbf{0} = (0,0,0)$. We show that there is no parametrization of S near $\mathbf{0}$ by contradiction. Suppose the contrary, then near $\mathbf{0}$, S is a graph. More precisely, there exists a neighbourhood $V \subseteq \mathbb{R}^3$ of $\mathbf{0}$ such that $S \cap V$ is equal to the graph of a function defined on a neighbourhood of $(0,0)$ in the yz-plane (case 1), or a function defined on a neighbourhood of $(0,0)$ in the xz-plane (case 2), or a function defined on a neighbourhood of $(0,0)$ in the xy-plane (case 3). We argue that none of these cases is possible (using MATH1050 stuff). Let B_ε be the open ball in $\mathbb{R}^2$ with centre $(0,0)$ and radius $\varepsilon > 0$.

(case 1) Note that for all (a fortiori, there is) $(y,z) \in B_\varepsilon \cap \{|z| > |y|\}$, there are two points of S which project down to (y,z). They are $(\pm\sqrt{z^2 - y^2}, y, z)$. Since any neighbourhood of $(0,0)$ contains a $B_\varepsilon \cap \{|z| > |y|\}$, it follows that this case is impossible.

(case 2) Similar to case 1.

(case 3) Note that for all (a fortiori, there is) $(x,y) \in B_\varepsilon - \{\mathbf{0}\}$, there are two points of S which project down to (x,y). They are $(x, y, \pm\sqrt{x^2 + y^2})$. Since any neighbourhood of $(0,0)$ contains a $B_\varepsilon - \{\mathbf{0}\}$, it follows that this case is impossible.

Solution 2. The following is a purely topological proof. We show that there is no open neighbourhood of $\mathbf{0}$ in S homeomorphic to $\mathbb{R}^2$. Suppose there is such neighbourhood, say U. Being homeomorphic to $\mathbb{R}^2$, we see that after removing $\mathbf{0}$ from U, U remains connected. However, being a neighbourhood of $\mathbf{0}$ in S, we see that after the removal, U is **not** connected! This follows from

$$U - \{\mathbf{0}\} = U \cap \{z > 0\} \sqcup U \cap \{z < 0\}$$

where each set in RHS is non-empty (for example, they contain $(\frac{3}{n}, \frac{4}{n}, \pm\frac{5}{n})$ for some large n).

6. (1 point) Let $p \in S$, then $\exists U \subseteq \mathbb{R}^2, V \subseteq S$ with V being a connected neighbourhood of p and a parametrization $X : U \to V$.
   Consider the function $f : U \to \mathbb{R} : (u, v) \mapsto \langle X(u, v) - p_0, X(u, v) - p_0\rangle$. We have

$$\frac{\partial f}{\partial u} = 2\Big\langle \frac{\partial X}{\partial u}, X - p_0\Big\rangle$$
$$\frac{\partial f}{\partial v} = 2\Big\langle \frac{\partial X}{\partial v}, X - p_0\Big\rangle.$$

   By assumption , $\frac{\partial f}{\partial u} = \frac{\partial f}{\partial v} = 0$. It follows that f is a constant function, which means that every point of V has a fixed distance, say r, from p_0. Since S is connected, the above is true for every point of the entire surface S. In other words, S is contained in a sphere with centre p_0 and radius r.

7. (1 point) Let y be a point in $\mathbb{R}^3$. Since S is compact, there is a point x_{max} in S such that

$$|y - x| \leqslant |y - x_{max}|$$

   for all $x \in S$.

   Lemma. The vector $y - x_{max}$ is orthogonal to S at x_{max}.
   Proof: Exercise. (Hint: See HW1 Q3 and HW2 Q6.)

   Now consider the continuous map $F : S \times S \to \mathbb{R} : (p, q) \mapsto |p - q|$. Since $S \times S$ is compact, F attains maximum at some point (p_0, q_0). By applying the lemma twice to $(y, x_{max}) = (p_0, q_0)$ and to $(y, x_{max}) = (q_0, p_0)$, we see that the vector $p_0 - q_0$ is orthogonal to S at p_0 and at q_0. Note that p_0 cannot be equal to q_0, for otherwise F is the zero map, and hence S is reduced to a point, in contradiction to the fact that S is a surface. Our desired straight line ℓ will then be the one passing through p_0 and q_0.

8. (1 point)
   <u>Proof that $S_i,\ i = 1, 2, 3$ is a surface:</u>
   It suffices to prove the result for S_1. Let $F : \mathbb{R}^3 \to \mathbb{R} : (x, y, z) \mapsto x^2 + y^2 + z^2 - ax$. Then $S_1 = F^{-1}(0)$. To prove that this subset is a surface, it is sufficient (WARNING: not necessary, see HW2 suggested ex. Q5) to show that 0 is a regular value of F. To see this, pick a point $(x_0, y_0, z_0) \in F^{-1}(0)$, and consider the differential $dF(x_0, y_0, z_0) : \mathbb{R}^3 \to \mathbb{R}$ whose matrix form is given by

$$dF(x_0, y_0, z_0) = [2x_0 - a\ 2y_0\ 2z_0].$$

   Suppose (x_0, y_0, z_0) is a critical point of F. Then the above matrix is zero, i.e. $x_0 = \frac{a}{2},\ y_0 = z_0 = 0$. But this is impossible, for otherwise, $0 = F(x_0, y_0, z_0) = (\frac{a}{2})^2 - a \cdot \frac{a}{2} = -\frac{a^2}{4}$, contradicting the fact that $a \neq 0$. Hence (x_0, y_0, z_0) is a regular point. We have shown that every point in $F^{-1}(0)$ is a regular point, hence 0 is a regular value of F, as desired.

Remark. Note that all the surfaces S_1, S_2 and S_3 are spheres. What are their centres and radii?

Proof that any pair of S_1, S_2 and S_3 intersect orthogonally:
It is sufficient to prove the result for the pair S_1 and S_2. We are going to show that for any $p = (x_0, y_0, z_0) \in S_1 \cap S_2$, the normal lines to the tangent planes T_pS_1 and T_pS_2 are orthogonal. By Tut. 4, we know that these normal lines are given by the gradients of the defining functions for S_1 and for S_2 respectively. According to the last part, they are

$$[2x_0 - a\ 2y_0\ 2z_0] \text{ and } [2x_0\ 2y_0 - b\ 2z_0]$$

respectively. Hence their inner product is

$$\begin{aligned}
& 2x_0(2x_0 - a) + 2y_0(2y_0 - b) + 4z_0^2 \\
= \ & 2(x_0^2 + y_0^2 + z_0^2 - ax_0) + 2(x_0^2 + y_0^2 + z_0^2 - by_0) \\
= \ & 0 \quad \because p \in S_1 \cap S_2.
\end{aligned}$$

Remark. An intuitive "proof" is the following: Observe that the origin $\mathbf{0}$ is an intersection point of S_1 and S_2, and the tangent planes are just the coordinate planes so that the orthogonality can be easily proved. For a general intersection point p, just argue that there is an isometry of $\mathbb{R}^3$ fixing S_1 and S_2, and sending $\mathbf{0}$ to p.