MATH 4030 Differential Geometry
Homework 1
Suggested solutions

1. (1 point) Let $\alpha : I \to \mathbb{R}$ be a curve
   Let $\phi : \mathbb{R}^3 \to \mathbb{R}^3$ be rigid motion, then
   $$\phi(x) = Ax + b \qquad \text{where } A \in O(3), b \in \mathbb{R}^3$$
   Since $A \in O(3)$, $|Ax| = |x|$ for any $x \in \mathbb{R}^3$ as
   $$\langle Ax, Ax\rangle = x^T A^T Ax = x^T x = \langle x, x\rangle$$
   $$\begin{aligned} L_a^b(\phi \circ \alpha) &= \int_a^b |(\phi \circ \alpha)'(t)|\, dt \\ &= \int_a^b |A\alpha'(t)|\, dt \\ &= \int_a^b |\alpha'(t)|\, dt \\ &= L_a^b(\alpha) \end{aligned}$$

2. (1 point)
   Solution 1.
   $$\begin{aligned} |\alpha(b) - \alpha(a)| &= \int_a^b \langle \alpha'(t), \frac{\alpha(b) - \alpha(a)}{|\alpha(b) - \alpha(a)|}\rangle\, dt && \text{Why do we write it in this way?} \\ &\leqslant \int_a^b |\alpha'(t)| \left|\frac{\alpha(b) - \alpha(a)}{|\alpha(b) - \alpha(a)|}\right|\, dt && \text{by Cauchy-Schwarz inequality} \\ &= \int_a^b |\alpha'(t)|\, dt \\ &= L_a^b(\alpha) \end{aligned}$$
   Solution 2.
   $$\begin{aligned} |\alpha(b) - \alpha(a)| &= \left|\int_a^b \alpha'(t)\ dt\right| \\ &\leqslant \int_a^b |\alpha'(t)|\, dt \\ &= L_a^b(\alpha) \end{aligned}$$
   Remark. Cauchy-Schwarz inequality has also been used in solution 2, where?

3. (1 point) Consider the function $|\alpha|^2$. Since $\alpha(t_0)$ is closest to the origin, we have $\frac{d}{dt}|\alpha|^2\Big|_{t=t_0} = 0$. (For this conclusion to be true, t_0 has to be in the *interior* of the domain but this is automatic since the domain is an open interval.) Hence we get $2\langle \alpha(t_0), \alpha'(t_0)\rangle = 0$. Geometrically speaking, that means the position vector $\alpha(t_0)$ is orthogonal to the velocity vector $\alpha'(t_0)$.

4. (1 point) Since ϕ is a diffeomorphism between two open intervals $I, J \subset \mathbb{R}$, ϕ is strictly monotone.
   If $\phi' > 0$, then $\phi(a) = c$ and $\phi(b) = d$, and we have

$$\begin{aligned} L_a^b(\alpha \circ \phi) &= \int_a^b |(\alpha \circ \phi)'(t)|\, dt \\ &= \int_a^b |\alpha'(\phi(t))\phi'(t)|\, dt \\ &= \int_a^b |\alpha'(\phi(t))|\phi'(t)\, dt \\ &= \int_c^d |\alpha'(\phi)|\, d\phi \\ &= L_c^d(\phi) \end{aligned}$$

   If $\phi' < 0$, then $\phi(a) = d$ and $\phi(b) = c$

$$\begin{aligned} L_a^b(\alpha \circ \phi) &= \int_a^b |(\alpha \circ \phi)'(t)|\, dt \\ &= \int_a^b |\alpha'(\phi(t))\phi'(t)|\, dt \\ &= \int_a^b |\alpha'(\phi(t))|(-\phi'(t))\, dt \\ &= -\int_d^c |\alpha'(\phi)|\, d\phi \\ &= L_c^d(\phi) \end{aligned}$$

5. (1 point)
   $\alpha(t) = (ae^{bt}\cos t, ae^{bt}\sin t)$
   $\alpha'(t) = (ae^{bt}(b\cos t - \sin t), ae^{bt}(b\sin t + \cos t))$
   $|\alpha'(t)| = ae^{bt}\sqrt{b^2+1}$

$$\begin{aligned} S(t) &= \int_{t_0}^t ae^{bu}\sqrt{b^2+1}\, du \\ &= \frac{a\sqrt{b^2+1}}{b}\left(e^{bt} - e^{bt_0}\right) \end{aligned}$$

   $t(s) = S^{-1}(s) = \frac{1}{b}\ln\left(\frac{bs}{a\sqrt{b^2+1}} + e^{bt_0}\right)$
   A reparametrization of α by arc length is therefore given by

$$\beta : \left(-\infty, \frac{a\sqrt{b^2+1}}{(-b)}e^{bt_0}\right) \to \mathbb{R}^2$$

$$\beta(s) = \left(a\left(\frac{bs}{a\sqrt{b^2+1}} + e^{bt_0}\right)\cos\left[\frac{1}{b}\ln\left(\frac{bs}{a\sqrt{b^2+1}} + e^{bt_0}\right)\right], a\left(\frac{bs}{a\sqrt{b^2+1}} + e^{bt_0}\right)\sin\left[\frac{1}{b}\ln\left(\frac{bs}{a\sqrt{b^2+1}} + e^{bt_0}\right)\right]\right)$$

6. (2 points) Let β be a reparametrization of α by arc length. Then $T(s) = \beta'(s) = \frac{\alpha'(t)}{|\alpha'(t)|}$.

$$\begin{aligned}
k_\alpha(t) &= \langle \frac{d}{ds} T(s), N(s) \rangle \\
&= \langle \frac{1}{|\alpha'(t)|} \frac{d}{dt} \left(\frac{\alpha'(t)}{|\alpha'(t)|} \right), N(t) \rangle \\
&= \langle \frac{1}{|\alpha'(t)|} \left(\frac{\alpha''(t)}{|\alpha'(t)|} - \frac{\alpha'(t)}{|\alpha'(t)|^2} \frac{d|\alpha'(t)|}{dt} \right), N(t) \rangle \\
&= \frac{\langle \alpha''(t), N(t) \rangle}{|\alpha'(t)|^2} \qquad (\because \langle \alpha'(t), N(t) \rangle = 0) \\
&= \frac{\det\left(\frac{\alpha'(t)}{|\alpha'(t)|}, \alpha''(t) \right)}{|\alpha'(t)|^2} \qquad (*) \\
&= \frac{\det\left(\alpha'(t), \alpha''(t) \right)}{|\alpha'(t)|^3}
\end{aligned}$$

$*$ If $\mathbf{u}$ and $\mathbf{v}$ are two planar vectors and J is the rotation about the origin by 90° anti-clockwise, then

$$\langle \mathbf{u}, J\mathbf{v} \rangle = -\det(\mathbf{u}, \mathbf{v}).$$

Prove it!

7. (2 points) The existence of such δ comes from the *uniform continuity theorem* in mathematical analysis: Let $f : [a, b] \to \mathbb{R}^k$ be a continuous function, then given any $\varepsilon > 0$ there exists $\delta > 0$ such that $(x, y \in [a, b]$ and $|x - y| < \delta)$ implies $|f(x) - f(y)| < \varepsilon$. Now given $\varepsilon > 0$. We apply this theorem to the pair $(f = \alpha', \frac{\varepsilon}{b-a})$ so that we get the corresponding $\delta > 0$. We show that this is the desired δ. Let $P = \{a = t_0 < t_1 < \cdots < t_n = b\}$ be a partition of $[a, b]$ such that $|t_i - t_{i-1}| < \delta$ for all i. For each $i = 1, 2, \ldots, n$ and each $t \in [t_{i-1}, t_i]$, applying mean value theorem to the real-valued function $x \mapsto \langle \alpha(x) - x \cdot \alpha'(t), [\alpha(t_i) - \alpha(t_{i-1})] - (t_i - t_{i-1})\alpha'(t) \rangle$ and using Cauchy-Schwarz inequality, we see that there is $c_i \in (t_{i-1}, t_i)$ such that $|[\alpha(t_i) - \alpha(t_{i-1})] - (t_i - t_{i-1})\alpha'(t)| \leqslant (t_i - t_{i-1})|\alpha'(c_i) - \alpha'(t)| < \frac{t_i - t_{i-1}}{b-a}\varepsilon$. Then

$$\begin{aligned}
&\left| L_a^b(\alpha, P) - \int_a^b |\alpha'(t)|\, dt \right| \\
= &\left| \sum_{i=1}^n \left[|\alpha(t_i) - \alpha(t_{i-1})| - \int_{t_{i-1}}^{t_i} |\alpha'(t)|\, dt \right] \right| \\
= &\left| \sum_{i=1}^n \frac{1}{t_i - t_{i-1}} \int_{t_{i-1}}^{t_i} [|\alpha(t_i) - \alpha(t_{i-1})| - (t_i - t_{i-1})|\alpha'(t)|]\, dt \right| \\
\leqslant &\sum_{i=1}^n \frac{1}{t_i - t_{i-1}} \int_{t_{i-1}}^{t_i} |[\alpha(t_i) - \alpha(t_{i-1})] - (t_i - t_{i-1})\alpha'(t)|\, dt \\
< &\sum_{i=1}^n \frac{1}{t_i - t_{i-1}} \int_{t_{i-1}}^{t_i} \frac{t_i - t_{i-1}}{b-a}\varepsilon\, dt \\
= &\ \varepsilon.
\end{aligned}$$

Finally, we show that $L_a^b(\alpha) = \sup\{L_a^b(\alpha, P) |\ P \text{ is a partition of } [a,b]\}$. Let A be the RHS. Let P, P' be two partitions of $[a,b]$. We say that P is *finer* than P', denoted $P \prec P'$, if P' is a subset of P. In other words, P is a partition of $[a,b]$ starting with P'. Note that the triangle inequality implies that if $P \prec P'$, then $L_a^b(\alpha, P') \leqslant L_a^b(\alpha, P) \leqslant A$. Observe also that if P' satisfies $|P'| < \delta$, then so does every P with $P \prec P'$. Now we prove the result by showing

$$\varepsilon > 0 \Longrightarrow |A - L_a^b(\alpha)| < \varepsilon.$$

Given $\varepsilon > 0$. By the definition of A, there is P such that $|A - L_a^b(\alpha, P)| < \varepsilon/2$ and by the first part of the problem we get a $\delta > 0$ such that $|P'| < \delta \Longrightarrow |L_a^b(\alpha, P') - L_a^b(\alpha)| < \varepsilon/2$. By further partitioning $[a,b]$, we get a $P' \prec P$ such that $|P'| < \delta$ so that $|L_a^b(\alpha, P') - L_a^b(\alpha)| < \varepsilon/2$. Note that $L_a^b(\alpha, P) \leqslant L_a^b(\alpha, P') \leqslant A \Longrightarrow |A - L_a^b(\alpha, P')| < \varepsilon/2$. Hence we have $\varepsilon > 0 \Longrightarrow |A - L_a^b(\alpha)| < \varepsilon$.

8. (1 point)
$\alpha(t) = (\sin t, \cos t + \log\tan\frac{t}{2})$
$\alpha'(t) = (\cos t, -\sin t + \csc t) = (\cos t, \frac{\cos^2 t}{\sin t})$
$\alpha'(t) = 0 \Longleftrightarrow \cos t = 0 \Longleftrightarrow t = \frac{\pi}{2}$
Hence α is regular except at $t = \frac{\pi}{2}$.
The square of the length of the segment in question is given by

$$\begin{aligned} & \sin^2 t \left[1 + \left(\frac{\frac{\cos^2 t}{\sin t}}{\cos t}\right)^2\right] \\ = \ & \sin^2 t(1 + \cot^2 t) \\ = \ & \sin^2 t + \cos^2 t \\ = \ & 1. \end{aligned}$$