THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH4030 Differential Geometry
Solution of Midterm

1.

$$\alpha'(t) = (2\sin^2 t, 2\sin t\cos t, -\sin t)$$
$$|\alpha'(t)| = \sqrt{5}\sin t$$

Let $t = t(s)$ be an increasing function of s with $t'(s) = \dfrac{1}{\sqrt{5}\sin t}$, then $\alpha(t(s))$ is p.b.a.l with respect to the parameter s.(Keep in mind that t is a function of s and s is also a function of t)

Then we can compute the Frenet frame,

$$\begin{aligned} T(s) &= \frac{\partial}{\partial s}\alpha(t(s)) \\ &= \alpha'(t)t'(s) \\ &= (2\sin^2 t, 2\sin t\cos t, -\sin t)\frac{1}{\sqrt{5}\sin t} \\ &= (\frac{2}{\sqrt{5}}\sin t, \frac{2}{\sqrt{5}}\cos t, -\frac{1}{\sqrt{5}}) \end{aligned}$$

$$\begin{aligned} N(s) &= \frac{T'(s)}{|T'(s)|} \\ &= \frac{T'(t)t'(s)}{|T'(t)t'(s)|} \\ &= \frac{T'(t)}{|T'(t)|} \\ &= \frac{(\frac{2}{\sqrt{5}}\cos t, -\frac{2}{\sqrt{5}}\sin t, 0)}{|(\frac{2}{\sqrt{5}}\cos t, -\frac{2}{\sqrt{5}}\sin t, 0)|} \\ &= (\cos t, -\sin t, 0) \end{aligned}$$

$$\begin{aligned} B(s) &= T(s)\times N(s) \\ &= (-\frac{1}{\sqrt{5}}\sin t, -\frac{1}{\sqrt{5}}\cos t, -\frac{2}{\sqrt{5}}) \end{aligned}$$

Then we can compute κ and τ

$$\begin{aligned} \kappa(s) &= |T'(s)| \\ &= |T'(t)t'(s)| \\ &= \frac{2}{\sqrt{5}}\frac{1}{\sqrt{5}\sin t} \\ &= \frac{2}{5\sin t} \end{aligned}$$

$$\begin{aligned}\tau(s) &= < B'(s), N(s) > \\ &= < B'(t)t'(s), N(s) > \\ &= \left\langle (-\frac{1}{\sqrt{5}}\cos t, \frac{1}{\sqrt{5}}\sin t, 0)\frac{1}{\sqrt{5}\sin t}, (\cos t, -\sin t, 0)\right\rangle \\ &= -\frac{1}{5\sin t}\end{aligned}$$

[Remark: if we want to compute κ and τ by the definitions, we must differentiate everything in terms of the arc-length parameter s, i.e, $\kappa \neq |T'(t)|$ and $\tau \neq < B'(t), N(t) >$]

We can also compute κ and τ using the formula we have derived for every regular space curve $\alpha(t)$

$$\kappa = \frac{|\alpha'(t) \times \alpha''(t)|}{|\alpha'(t)|^3}$$

$$\tau = -\frac{< \alpha'(t) \times \alpha''(t), \alpha'''(t) >}{|\alpha'(t) \times \alpha''(t)|^2}$$

In this question,
$\alpha'(t) = (2\sin^2 t, 2\sin t\cos t, -\sin t)$
$\alpha''(t) = (4\sin t\cos t, 2\cos^2 t - 2\sin^2 t, -\cos t)$
$\alpha'''(t) = (4\cos^2 t - 4\sin^2 t, -8\sin t\cos t, \sin t)$
$\alpha'(t) \times \alpha''(t) = -2\sin^2 t(\sin t, \cos t, 2)$
So we have

$$\begin{aligned}\kappa &= \frac{|\alpha'(t) \times \alpha''(t)|}{|\alpha'(t)|^3} \\ &= \frac{2\sqrt{5}\sin^2 t}{5\sqrt{5}\sin^3 t} \\ &= \frac{2}{5\sin t}\end{aligned}$$

$$\begin{aligned}\tau &= -\frac{< \alpha'(t) \times \alpha''(t), \alpha'''(t) >}{|\alpha'(t) \times \alpha''(t)|^2} \\ &= -\frac{4\sin^3 t}{20\sin^4 t} \\ &= -\frac{1}{5\sin t}\end{aligned}$$

2. By the assumption, we have $\alpha^{'}(s) \perp (\alpha(s) - p_0)$ for any $s \in I$,

$$< \alpha^{'}(s), \alpha(s) - p_0 >= 0$$
$$< \alpha(s) - p_0, \alpha(s) - p_0 >^{'} = 0$$

So $|\alpha(s) - p_0| = R$ for some constant R. Obviously, $R \geq 0$.
If $R = 0$, then $\alpha(s) = p_0$ for any $s \in I$ and $\alpha^{'}(s) = 0$. This is a contraction to $\alpha(s)$ is p.b.a.l.
So $R > 0$ and $|\alpha(s) - p_0| = R$ for any $s \in I$.
This means α lies on the circle centered at p_0 with radius R.

3. (a) $\nabla F = (-y, -x, 1) \neq 0$ for any (x, y, z).
So 0 is a regular value of F.
Then $F^{-1}(0) = \{(x, y, z) \in \mathbb{R}^3 : z = xy\}$ is a regular surface.

(b)

$$X_u = (1, 0, v)$$
$$X_v = (0, 1, u)$$

So we have the first fundamental form

$$g_{ij} = \begin{bmatrix} 1 + v^2 & uv \\ uv & 1 + u^2 \end{bmatrix}$$

Then we compute the second fundamental form

$$X_u \times X_v = (-v, -u, 1)$$
$$N = \frac{(-v, -u, 1)}{\sqrt{1 + u^2 + v^2}}$$
$$X_{uu} = (0, 0, 0)$$
$$X_{uv} = (0, 0, 1)$$
$$X_{vv} = (0, 0, 0)$$

So the second fundamental form is

$$A_{ij} = \begin{bmatrix} 0 & \frac{1}{\sqrt{1 + u^2 + v^2}} \\ \frac{1}{\sqrt{1 + u^2 + v^2}} & 0 \end{bmatrix}$$

Then we can compute the Gauss curvature and mean curvature

$$\begin{aligned} K &= \frac{det(A)}{det(g)} \\ &= -\frac{1}{1 + u^2 + v^2} \frac{1}{1 + u^2 + v^2} \\ &= -\frac{1}{[1 + u^2 + v^2]^2} \end{aligned}$$

$$\begin{aligned} H &= tr(g^{-1}A) \\ &= tr\left(\frac{1}{1 + u^2 + v^2} \begin{bmatrix} 1 + u^2 & -uv \\ -uv & 1 + v^2 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{\sqrt{1 + u^2 + v^2}} \\ \frac{1}{\sqrt{1 + u^2 + v^2}} & 0 \end{bmatrix} \right) \\ &= -\frac{2uv}{[1 + u^2 + v^2]^{\frac{3}{2}}} \end{aligned}$$

4. The surface is the part of the unit sphere which lies between $z = 0$ and $z = \frac{\sqrt{2}}{2}$.

$$X_u = (\cos u \cos v, \cos u \sin v, -\sin u)$$
$$X_v = (-\sin u \sin v, \sin u \cos v, 0)$$

The first fundamental form is $g_{ij} = \begin{bmatrix} 1 & 0 \\ 0 & \sin^2 u \end{bmatrix}$

Since X is inject, the area is

$$\begin{aligned} Area &= \int_0^{2\pi} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sqrt{det(g)} du dv \\ &= \int_0^{2\pi} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin u du dv \\ &= \int_0^{2\pi} \frac{\sqrt{2}}{2} dv \\ &= \sqrt{2}\pi \end{aligned}$$

5. For any $p \in S$, let $X : U \to S$ be a parametrization of S around p.
   And $X(u,v) = (X_1(u,v), X_2(u,v), X_3(u,v))$ where X_1, X_2, X_3 are smooth functions on U.
   Then $(f \circ X)(u,v) = (-X_1(u,v), -X_2(u,v), -X_3(u,v))$ is smooth since X_1, X_2, X_3 are smooth.
   So f is a smooth map.
   For any $p \in S$, $\vec{v} \in T_pS$, let $\alpha(t) : (-\epsilon, \epsilon) \to S$ be any regular curve on S with $\alpha(0) = p$ and $\alpha^{'}(0) = \vec{v}$.
   Then by the definition,

$$\begin{aligned} df_p(\vec{v}) &= \frac{d}{dt}\Big|_{t=0} f(\alpha(t)) \\ &= \frac{d}{dt}\Big|_{t=0} - \alpha(t) \\ &= -\alpha^{'}(0) \\ &= -\vec{v} \end{aligned}$$