THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH4030 Differential Geometry
Solution of Assignment 6

1. We need to show that S is totally umbilical, thus by the lecture notes S is contained in a plane or a sphere.
For any $p \in S$, let $\{\kappa_1, \kappa_2\}$, $\{v_1, v_2\}$ be the corresponding eigenvalues, eigenvectors of the shape operator and $\{v_1, v_2\}$ forms an orthonormal basis of T_pS. The aim is to show that $\kappa_1 = \kappa_2$.

case1 $\kappa_1 = \kappa_2 = 0$
Then p is an umbilical point.

case2 They are not both zero, i.e,
$$\kappa_1^2 + \kappa_2^2 \neq 0$$
For any unit vector $v \in T_pS$, $v = v_1 \cos\theta + v_2 \sin\theta$ for some θ.
Let $\alpha(t) : (-\epsilon, \epsilon) \to S$ be a geodesic with $\alpha(0) = p$ and $\alpha'(0) = v$. We have
$$\begin{aligned}\alpha''(t) &= D_{\alpha'(t)}\alpha'(t) \\ &= (D_{\alpha'(t)}\alpha'(t))^\top + (D_{\alpha'(t)}\alpha'(t))^\perp \\ &=< D_{\alpha'(t)}\alpha'(t), N > N + \nabla_{\alpha'(t)}\alpha'(t) \\ &=< \alpha'(t), -dN(\alpha'(t)) > +0\end{aligned}$$
Thus at $t = 0$,
$$\begin{aligned}\alpha''(0) &=< \alpha'(0), -dN(\alpha'(0)) > \\ &=< v_1 \cos\theta + v_2 \sin\theta, -dN(v_1 \cos\theta + v_2 \sin\theta) > \\ &=< v_1 \cos\theta + v_2 \sin\theta, \kappa_1 v_1 \cos\theta + \kappa_2 v_2 \sin\theta > \\ &= \kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta\end{aligned}$$
By the assumption, κ_1 and κ_2 are not both zero. So $\alpha''(0) = 0$ has at most four roots in $[0, 2\pi]$. For the other θ, $\alpha''(0) \neq 0$ and $\alpha''(t) =< D_{\alpha'(t)}\alpha'(t), N > N$ by the previous computation. Let $\{T_\alpha, N_\alpha, B_\alpha\}$ be the Frenet frame of α as a space curve, then N_α is parallel to N. By choosing the direction of N, we may assume $N_\alpha = N$.
Since all geodesics are plane curves, so $\tau = 0$,
$$N_\alpha'(t) = -\kappa T_\alpha - \tau B_\alpha = -\kappa\alpha'(t)$$
So we have
$$\begin{aligned}dN(v) &= dN_\alpha(v) \\ &= N_\alpha'(0) \\ &= -\kappa v\end{aligned}$$
Which means
$$-dN(v_1 \cos\theta + v_2 \sin\theta) = \kappa(v_1 \cos\theta + v_2 \sin\theta)$$
$$\kappa_1 v_1 \cos\theta + \kappa_2 v_2 \sin\theta = \kappa v_1 \cos\theta + \kappa v_2 \sin\theta$$
for almost all θ. Then we have $\kappa_1 = \kappa_2$

2. Suppose NOT. Then there are α and β which are simple closed geodesics in S satisfy

$$\alpha \cap \beta = \phi$$

Since S is topologically a sphere, α and β divide S into three parts D_1, D_2 and Σ where D_1, D_2 are topologically disks and Σ is topologically a cylinder.
Applying Gauss-Bonnet theorem to Σ,

$$\int_\Sigma K dA \pm \int_\alpha \kappa_g ds \pm \int_\beta \kappa_g ds = 2\pi\chi(\Sigma) = 0$$

$$\int_\Sigma K dA = 0$$

This is a contraction to $K > 0$.

3. (a) $[K > 0]$
From the assumption that Σ is closed, we have that it is compact without boundary.
By Q5(b) in hw5, we know that there must be a point $p_1 \in \Sigma$ such that

$$K(p_1) > 0$$

(b) $[K < 0]$
By the assumption that Σ is not homeomorphic to a sphere, we have $\chi(\Sigma) = 2 - 2g \leq 2 - 2 = 0$.
By the Gauss-Bonnet theorem,

$$\int_\Sigma K dA = 2\pi\chi(\Sigma) \leqslant 0$$

So there must be a point $p_2 \in \Sigma$ such that

$$K(p_2) < 0$$

(c) $[K = 0]$
W.L.O.G, we may assume Σ is connected. Otherwise, we consider each connected component of Σ in (a) and (b).
Let $\gamma(t)$ be any path joining p_1 and p_2, then $K(\gamma(t))$ is a smooth function on t. By the intermediate value theorem, there is a point $p_3 \in \Sigma$ such that

$$K(p_3) = 0$$

4.

$$X(u,v) = ((2+\cos u)\cos v, (2+\cos u)\sin v, \sin u)$$
$$X_u = (-\sin u \cos v, -\sin u \sin v, \cos u)$$
$$X_v = (-(2+\cos u)\sin v, (2+\cos u)\cos v, 0)$$

The first fundamental form is

$$g_{ij} = \begin{bmatrix} 1 & 0 \\ 0 & (2+\cos u)^2 \end{bmatrix}$$

Use the formula of Q3 in hw5,

$$\begin{aligned} K &= -\frac{1}{2\sqrt{EG}}\left[\left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{EG}}\right)_u\right] \\ &= -\frac{1}{2\sqrt{(2+\cos u)^2}}\left[\left(\frac{0}{\sqrt{EG}}\right)_v + \left(\frac{-2(2+\cos u)\sin u}{\sqrt{(2+\cos u)^2}}\right)_u\right] \\ &= -\frac{1}{2(2+\cos u)}\left[-2\cos u\right] \\ &= \frac{\cos u}{2+\cos u} \end{aligned}$$

So the integration is

$$\begin{aligned} \int_T K dA &= \int_0^{2\pi}\int_0^{2\pi} \frac{\cos u}{(2+\cos u)} \cdot (2+\cos u) du dv \\ &= \int_0^{2\pi}\int_0^{2\pi} \cos u du dv \\ &= 0 \end{aligned}$$

5. Let α, β, γ be arc-length parametrization of p_0q, qr, rp_0.
Since p_0q, rp_0 are meridians and qr lies on the equator, α, β, γ are all geodesics.
Let $\vec{V}$ be the path of the parallel transport of v. Then $\vec{V}(\alpha(p_0)) = v = \alpha^{'}(p_0)$.

$$\nabla_{\alpha'}\alpha^{'} = 0$$

$$\vec{V}(\alpha) = \alpha^{'}$$

$$\vec{V}(\alpha(q)) = \alpha^{'}(q)$$

Since $\beta^{'}$, $\vec{V}$ are both parallel vector fields on β and $\beta^{'}(q) \perp \vec{V}(q)$, we have

$$\vec{V}(r) = -\gamma^{'}(r)$$

Similarly to p_0q, on rp_0

$$\vec{V}(\gamma) = -\gamma^{'}$$

So the angle between the final position of v and the initial vector v is θ.