THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH4030 Differential Geometry
Solution of Assignment 4

1. (a) $S_1 = \{(x, y, x^2 + y^2)\}$
$X_x = (1, 0, 2x)$
$X_y = (0, 1, 2y)$

The first fundamental form is

$$g = \begin{bmatrix} 1+4x^2 & 4xy \\ 4xy & 1+4y^2 \end{bmatrix}$$
$$g^{-1} = \frac{1}{1+4x^2+4y^2} \begin{bmatrix} 1+4y^2 & -4xy \\ -4xy & 1+4x^2 \end{bmatrix}$$

Then we compute the second fundamental form,
$N = \frac{X_x \times X_y}{|X_x \times X_y|} = \frac{1}{\sqrt{1+4x^2+4y^2}}(-2x, -2y, 1)$
$X_{xx} = (0, 0, 2)$
$X_{xy} = (0, 0, 0)$
$X_{yy} = (0, 0, 2)$
$A = \frac{2}{\sqrt{1+4x^2+4y^2}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

The shape operator is
$S = g^{-1}A = \frac{2}{[1+4x^2+4y^2]^{\frac{3}{2}}} \begin{bmatrix} 1+4y^2 & -4xy \\ -4xy & 1+4x^2 \end{bmatrix}$

The curvatures are
$K = det(S) = \frac{4}{[1+4x^2+4y^2]^2}$

$H = tr(S) = \frac{4(1+2x^2+2y^2)}{[1+4x^2+4y^2]^{\frac{3}{2}}}$

For $p = (0, 0, 0)$,
$A(p) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$
$S(p) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$
So p is an umbilical point. The principle curvature is 2 and all unit vectors in T_pS_1 are the principle direction.

(b) $S_2 = \{(x, y, x^2 - y^2)\}$
$X_x = (1, 0, 2x)$
$X_y = (0, 1, -2y)$

The first fundamental form is

$$g = \begin{bmatrix} 1+4x^2 & -4xy \\ -4xy & 1+4y^2 \end{bmatrix}$$
$$g^{-1} = \frac{1}{1+4x^2+4y^2} \begin{bmatrix} 1+4y^2 & 4xy \\ 4xy & 1+4x^2 \end{bmatrix}$$

Then we compute the second fundamental form,
$N = \dfrac{X_x \times X_y}{|X_x \times X_y|} = \dfrac{1}{\sqrt{1+4x^2+4y^2}}(-2x, 2y, 1)$
$X_{xx} = (0, 0, 2)$
$X_{xy} = (0, 0, 0)$
$X_{yy} = (0, 0, -2)$
$A = \dfrac{2}{\sqrt{1+4x^2+4y^2}} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

The shape operator is
$S = g^{-1}A = \dfrac{2}{[1+4x^2+4y^2]^{\frac{3}{2}}} \begin{bmatrix} 1+4y^2 & -4xy \\ 4xy & -(1+4x^2) \end{bmatrix}$

The curvatures are
$K = det(S) = -\dfrac{4}{[1+4x^2+4y^2]^2}$

$H = tr(S) = \dfrac{8(y^2 - x^2)}{[1+4x^2+4y^2]^{\frac{3}{2}}}$

For $p = (0, 0, 0)$,
$A(p) = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$
$S(p) = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$
The principle curvatures are $\{2, -2\}$ and the principle directions are $\{\partial_x, \partial_y\}$

2.

$$X = (\cosh v \cos u, \cosh v \sin u, v)$$
$$X_u = (-\cosh v \sin u, \cosh v \cos u, 0)$$
$$X_v = (\sinh v \cos u, \sinh v \sin u, 1)$$

The first fundamental form is

$$g = \begin{bmatrix} \cosh^2 v & 0 \\ 0 & \cosh^2 v \end{bmatrix}$$
$$g^{-1} = \frac{1}{\cosh^2 v} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Then we compute the second fundamental form,

$N = \frac{X_u \times X_v}{|X_y \times X_v|} = \frac{1}{\cosh v}(\cos u, \sin u, -\sinh v)$

$X_{uu} = (-\cosh v \cos u, -\cosh v \sin u, 0)$

$X_{uv} = (-\sinh v \sin u, \sinh v \cos u, 0)$

$X_{vv} = (\cosh v \cos u, \cosh v \sin u, 0)$

$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

The shape operator is

$S = g^{-1}A = \frac{1}{\cosh^2 v} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

The curvatures are

$K = det(S) = -\frac{1}{\cosh^4 v}$

$H = tr(S) = 0$

3. Let $X : (-\frac{\pi}{2}, \frac{\pi}{2}) \times \mathbb{R} \to S$ to be

$$X(u,v) = (a\cos u \cos v, b \cos u \sin v, c \sin u)$$

Then X gives a parametrization of $S \setminus \{(0,0,\pm c)\}$. $u \neq \pm\frac{\pi}{2}$ because $X_v(\pm\frac{\pi}{2}, v) = 0$.

$$X_u = (-a \sin u \cos v, -b \sin u \sin v, c \cos u)$$
$$X_v = (-a \cos u \sin v, b \cos u \cos v, 0)$$

The first fundamental form is

$$g = \begin{bmatrix} (a^2\cos^2 v + b^2 \sin^2 v)\sin^2 u + c^2\cos^2 u & (a^2 - b^2)\sin u \cos u \sin v \cos v \\ (a^2 - b^2)\sin u \cos u \sin v \cos v & (a^2 \sin^2 v + b^2 \cos^2 v)\cos^2 u \end{bmatrix}$$

$$\begin{aligned} g^{-1} = &\frac{1}{[a^2b^2\sin^2 u + c^2(a^2\sin^2 v + b^2\cos^2 v)\cos^2 u]\cos^2 u} \\ &\cdot \begin{bmatrix} (a^2\sin^2 v + b^2\cos^2 v)\cos^2 u & -(a^2 - b^2)\sin u \cos u \sin v \cos v \\ -(a^2 - b^2)\sin u \cos u \sin v \cos v & (a^2\cos^2 v + b^2 \sin^2 v)\sin^2 u + c^2 \cos^2 u \end{bmatrix} \end{aligned}$$

Then we compute the second fundamental form,

$$N = \frac{X_u \times X_v}{|X_y \times X_v|} = -\frac{(bc \cos u \cos v, ac \cos u \sin v, ab \sin u)}{\sqrt{a^2b^2\sin^2 u + c^2(a^2\sin^2 v + b^2\cos^2 v)\cos^2 u}}$$
$$X_{uu} = (-a\cos u \cos v, -b\cos u \sin v, -c \sin u)$$
$$X_{uv} = (a \sin u \sin v, -b \sin u \cos v, 0)$$
$$X_{vv} = (-a \cos u \cos v, -b \cos u \sin v, 0)$$
$$A = \frac{abc}{\sqrt{a^2b^2\sin^2 u + c^2(a^2\sin^2 v + b^2\cos^2 v)\cos^2 u}} \begin{bmatrix} 1 & 0 \\ 0 & \cos^2 u \end{bmatrix}$$

The shape operator is

$$\begin{aligned} S &= g^{-1}A \\ &= \frac{abc}{[a^2b^2\sin^2 u + c^2(a^2\sin^2 v + b^2\cos^2 v)\cos^2 u]^{\frac{3}{2}}\cos^2 u} \\ &\cdot \begin{bmatrix} (a^2\sin^2 v + b^2\cos^2 v)\cos^2 u & -(a^2 - b^2)\sin u \cos^3 u \sin v \cos v \\ -(a^2 - b^2)\sin u \cos u \sin v \cos v & [(a^2\cos^2 v + b^2\sin^2 v)\sin^2 u + c^2\cos^2 u]\cos^2 u \end{bmatrix} \end{aligned}$$

The curvatures are

$$
\begin{aligned}
K &= det(S) \\
&= \frac{det(A)}{det(g)} \\
&= \frac{a^2b^2c^2\cos^2 u}{a^2b^2\sin^2 u + c^2(a^2\sin^2 v + b^2\cos^2 v)\cos^2 u} \\
&\quad \cdot \frac{1}{[a^2b^2\sin^2 u + c^2(a^2\sin^2 v + b^2\cos^2 v)\cos^2 u]\cos^2 u} \\
&= \frac{a^2b^2c^2}{[a^2b^2\sin^2 u + c^2(a^2\sin^2 v + b^2\cos^2 v)\cos^2 u]^2}
\end{aligned}
$$

$$
\begin{aligned}
H &= tr(S) \\
&= \frac{abc\{(a^2\sin^2 v + b^2\cos^2 v)\cos^2 u + [(a^2\cos^2 v + b^2\sin^2 v)\sin^2 u + c^2\cos^2 u]\cos^2 u\}}{[a^2b^2\sin^2 u + c^2(a^2\sin^2 v + b^2\cos^2 v)\cos^2 u]^{\frac{3}{2}}\cos^2 u} \\
&= \frac{abc\{a^2\sin^2 v + b^2\cos^2 v + (a^2\cos^2 v + b^2\sin^2 v)\sin^2 u + c^2\cos^2 u\}}{[a^2b^2\sin^2 u + c^2(a^2\sin^2 v + b^2\cos^2 v)\cos^2 u]^{\frac{3}{2}}}
\end{aligned}
$$

Let $f(u,v)$ be the function in the denominator of the expressions of K and H, then

$$
\begin{aligned}
f(u,v) &= a^2b^2\sin^2 u + c^2(a^2\sin^2 v + b^2\cos^2 v)\cos^2 u \\
&= a^2b^2(1-\cos^2 u) + c^2[a^2(1-\cos^2 v) + b^2\cos^2 v]\cos^2 u \\
&= a^2b^2 - a^2b^2\cos^2 u + a^2c^2\cos^2 u - a^2c^2\cos^2 u\cos^2 v + b^2c^2\cos^2 u\cos^2 v \\
&= a^2b^2 + a^2(c^2-b^2)\cos^2 u + c^2(b^2-a^2)\cos^2 u\cos^2 v
\end{aligned}
$$

By the assumption $a > b > c > 0$, then f attains its minimum when

$$\cos^2 u = \cos^2 v = 1$$

and $f_{min} = b^2c^2 > 0$
So K attains its maximum at $(\underline{+}a, 0, 0)$.

[claim: H attains its maximum at $(\underline{+}a, 0, 0)$]

$$
\begin{aligned}
H &= \frac{abc\{a^2\sin^2 v + b^2\cos^2 v + (a^2\cos^2 v + b^2\sin^2 v)\sin^2 u + c^2\cos^2 u\}}{[a^2b^2\sin^2 u + c^2(a^2\sin^2 v + b^2\cos^2 v)\cos^2 u]^{\frac{3}{2}}} \\
&= \frac{abc\{a^2(1-\cos^2 v) + b^2\cos^2 v + [a^2\cos^2 v + b^2(1-\cos^2 v)](1-\cos^2 u) + c^2\cos^2 u\}}{f^{\frac{3}{2}}} \\
&\qquad \textit{change everything into } \cos \\
&= \frac{abc\{a^2 + b^2 + (c^2-b^2)\cos^2 u + (b^2-a^2)\cos^2 u\cos^2 v\}}{f^{\frac{3}{2}}} \\
&= \frac{abc}{\sqrt{f}} \cdot \frac{a^2 + b^2 + (c^2-b^2)\cos^2 u + (b^2-a^2)\cos^2 u\cos^2 v}{f}
\end{aligned}
$$

Since $\dfrac{abc}{\sqrt{f}}$ attains its maximum at $(\pm a, 0, 0)$, we only need to show that the second term also attains its maximum at $(\pm a, 0, 0)$.
At $(\pm a, 0, 0)$, $\cos^2 u = \cos^2 v = 1$, the second term is $\dfrac{b^2+c^2}{b^2c^2}$.

$$\begin{aligned}
&\frac{a^2+b^2+(c^2-b^2)\cos^2 u+(b^2-a^2)\cos^2 u\cos^2 v}{f} \le \frac{b^2+c^2}{b^2c^2}\\
\Leftrightarrow&\frac{a^2+b^2+(c^2-b^2)\cos^2 u+(b^2-a^2)\cos^2 u\cos^2 v}{a^2b^2+a^2(c^2-b^2)\cos^2 u+c^2(b^2-a^2)\cos^2 u\cos^2 v} \le \frac{b^2+c^2}{b^2c^2}\\
\Leftrightarrow&\{a^2+b^2+(c^2-b^2)\cos^2 u+(b^2-a^2)\cos^2 u\cos^2 v\}b^2c^2\\
&\le \{a^2b^2+a^2(c^2-b^2)\cos^2 u+c^2(b^2-a^2)\cos^2 u\cos^2 v\}(b^2+c^2)\\
\Leftrightarrow&\{a^2b^2c^2+b^4c^2+b^2c^2(c^2-b^2)\cos^2 u+b^2c^2(b^2-a^2)\cos^2 u\cos^2 v\}\\
&\le \{a^2b^2c^2+a^2b^4+a^2(c^2-b^2)(c^2+b^2)\cos^2 u+c^2(b^2-a^2)(c^2+b^2)\cos^2 u\cos^2 v\}\\
\Leftrightarrow&(b^4c^2-a^2b^4)+(c^2-b^2)(b^2c^2-a^2b^2-a^2c^2)\cos^2 u+(b^2-a^2)(-c^4)\cos^2 u\cos^2 v \le 0
\end{aligned}$$

By the assumption $a > b > c > 0$, we have $(c^2-b^2)(b^2c^2-a^2b^2-a^2c^2) > 0$ and $(b^2-a^2)(-c^4) > 0$, so

$$\begin{aligned}
&(b^4c^2-a^2b^4)+(c^2-b^2)(b^2c^2-a^2b^2-a^2c^2)\cos^2 u+(b^2-a^2)(-c^4)\cos^2 u\cos^2 v\\
&\le (b^4c^2-a^2b^4)+(c^2-b^2)(b^2c^2-a^2b^2-a^2c^2)+(b^2-a^2)(-c^4)\\
&= b^4c^2-a^2b^4+b^2c^4-b^4c^2-a^2b^2c^2+a^2b^4-a^2c^4+a^2b^2c^2-b^2c^4+a^2c^4\\
&= 0
\end{aligned}$$

$'='$ holds if and only if $\cos^2 u = \cos^2 v = 1$.
This completes the proof.

4. Let $v_1 = (1, 0, 0), v_2 = (0, 1, 0)$. Then $span\{v_1, v_2\} = T_pS_1$ for any $p \in S_1$.

$$df(v_1) = f_x = (-\sin x, \cos x, 0)$$
$$df(v_2) = f_y = (0, 0, 1)$$

So we have

$$< df(v_i), df(v_j) >= \delta_{ij} =< v_i, v_j >$$

Then by the definition, f is a local isometry.

5. Let $X : U \to S_1$ be a parametrization of S_1.
   Then $f \circ X : U \to S_2$ is a parametrization of S_2 except at a set of measure zero.
   By the assumption that f is an isometry, so

$$< \partial_i(f \circ X), \partial_j(f \circ X) >=< df(X_i), df(X_j) >=< X_i, X_j >$$

   Then $\sqrt{det(g_1)} = \sqrt{det(g_2)}$ where g_1, g_2 are the first fundamental forms of S_1, S_2.

$$\begin{aligned} Area(S_1) &= \int_U \sqrt{det(g_1)} dA \\ &= \int_U \sqrt{det(g_2)} dA \\ &= Area(S_2) + 0 \\ &= Area(S_2) \end{aligned}$$

6.

$$X(x,y) = (x, y, f(x,y))$$
$$X_x = (1, 0, f_x)$$
$$X_y = (0, 1, f_y)$$

The first fundamental form is

$$g = \begin{bmatrix} 1+f_x^2 & f_x f_y \\ f_x f_y & 1+f_y^2 \end{bmatrix}$$
$$g^{-1} = \frac{1}{1+f_x^2+f_y^2} \begin{bmatrix} 1+f_y^2 & -f_x f_y \\ -f_x f_y & 1+f_x^2 \end{bmatrix}$$

Then we compute the second fundamental form,

$N = \dfrac{X_x \times X_y}{|X_x \times X_y|} = \dfrac{1}{\sqrt{1+f_x^2+f_y^2}}(-f_x, -f_y, 1)$

$X_{xx} = (0, 0, f_{xx})$

$X_{xy} = (0, 0, f_{xy})$

$X_{yy} = (0, 0, f_{yy})$

$A = \dfrac{1}{\sqrt{1+f_x^2+f_y^2}} \begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix}$

The mean curvature is

$$\begin{aligned} H &= tr(g^{-1}A) \\ &= \frac{(1+f_x^2)f_{yy} - 2f_x f_y f_{xy} + (1+f_y^2)f_{xx}}{[1+f_x^2+f_y^2]^{\frac{3}{2}}} \end{aligned}$$

So

$$H = 0 \Leftrightarrow (1+f_x^2)f_{yy} - 2f_x f_y f_{xy} + (1+f_y^2)f_{xx} = 0$$

7. Let $X : (0, \infty) \times (0, \pi) \to S_1$ be

$$X(r, \theta) = (0, r\sin\theta, r\cos\theta)$$

$$X_r = (0, \sin\theta, \cos\theta)$$
$$X_\theta = (0, r\cos\theta, -r\sin\theta)$$
$$g_1 = \begin{bmatrix} 1 & 0 \\ 0 & r^2 \end{bmatrix}$$

Let $Y : (0, \infty) \times (0, \pi) \to S_2$ be

$$Y(r, \theta) = (\frac{r}{\sqrt{2}}\cos(\sqrt{2}\theta), \frac{r}{\sqrt{2}}\sin(\sqrt{2}\theta), \frac{r}{\sqrt{2}})$$

$$Y_r = (\frac{1}{\sqrt{2}}\cos(\sqrt{2}\theta), \frac{1}{\sqrt{2}}\sin(\sqrt{2}\theta), \frac{1}{\sqrt{2}})$$
$$Y_\theta = (-r\sin(\sqrt{2}\theta), r\cos(\sqrt{2}\theta), 0)$$
$$g_2 = \begin{bmatrix} 1 & 0 \\ 0 & r^2 \end{bmatrix}$$

Then we have $f = Y \circ X^{-1} : S_1 \to S_2$ is a differentiable map and

$$df(X_r) = Y_r$$

$$df(X_\theta) = Y_\theta$$

Since $g_1 = g_2$, f is a local isometry.

8. (a) Let $G = \{f : S \to S \big| f, f^{-1}\text{are both differentiable}, f\text{is a local isometry}\}$.
1. Obviously, the identity map is in G.
2. Let $f \in G$, then f^{-1} exists and is differentiable. Let $v_1, v_2 \in T_pS$,

$$\begin{aligned} < (d(f^{-1}))(v_1), (d(f^{-1}))(v_2) > &=< df(d(f^{-1})(v_1)), df(d(f^{-1})(v_2)) > \\ &=< v_1, v_2 > \end{aligned}$$

So f^{-1} is also a local isometry.
3. Let $f, g \in G$, then for any $v_1, v_2 \in T_pS$

$$\begin{aligned} < (d(f \circ g))(v_1), (d(f \circ g))(v_2) > &=< df(dg(v_1)), df(dg(v_2)) > \\ &=< dg(v_1), dg(v_2) > \\ &=< v_1, v_2 > \end{aligned}$$

So $f \circ g \in G$ since $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$ exists.
4. By the definition of composition, $(f \circ g) \circ h = f \circ (g \circ h)$.
So G is a group under composition.

(b) The isometry group is $span\{\text{rotation}, \text{reflection}\}$ under composition.