THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH4030 Differential Geometry
Solution of Assignment 3

1. (a)

$$
\begin{aligned}
X(u,v) &= (f(u)\cos v, f(u)\sin v, g(u)) \\
X_u &= (f^{'}(u)\cos v, f^{'}(u)\sin v, g^{'}(u)) \\
X_v &= (-f(u)\sin v, f(u)\cos v, 0)
\end{aligned}
$$

$\{X_u, X_v\}$ is linearly independent since $g^{'}(u) > 0$.
So the image S is a surface.

(b)

$$X_u \times X_v = (-fg^{'}\cos v, -fg^{'}\sin v, ff^{'})$$

So the equation of the normal line is

$$
\begin{aligned}
\alpha(t) &= X(u,v) + tX_u \times X_v \\
&= (f\cos v(1-tg^{'}), f\sin v(1-tg^{'}), g - tff^{'})
\end{aligned}
$$

When $t = \dfrac{1}{g^{'}}$, $\alpha(t) = (0, 0, g - \dfrac{ff^{'}}{g^{'}})$.
So the normal lines of S pass through the z-axis.

2. Let $x_0 \in S$, for any $y_0 \in S$
   S is connected, so there is a smooth curve $\alpha(s) : [0, l] \to S$ such that

   $$\alpha(0) = x_0, \alpha(l) = y_0$$

   So $\alpha^{'}(s) \in T_{\alpha(s)}S$ for any s.
   By the assumption, all normal lines of S pass through p_0,

   $$(p_0 - \alpha(s)) \perp T_{\alpha(s)}S$$

   $$< p_0 - \alpha(s), \alpha^{'}(s) >= 0$$

   $$< p_0 - \alpha(s), p_0 - \alpha(s) >^{'}= 0$$

   So $|\alpha(s) - p_0| = R$ for some positive constant R. ($R > 0$ because $\alpha(s)$ contains not only one point)

   $$|x_0 - p_0| = |y_0 - p_0| = R$$

   So S lies on a sphere since y_0 is arbitrary.

3. Let $A = sup\{|x - y| : x, y \in S\}$.
   S is compact, so there are $x_0, y_0 \in S$ such that

$$|x_0 - y_0| = A$$

   Let l be the straight line joining x_0 and y_0.
   Let $\alpha(s) : (-\epsilon, \epsilon) \to S$ be any regular curve in S with $\alpha(0) = x_0$.
   So $< \alpha(s) - y_0, \alpha(s) - y_0 >$ attains a maximum at $t = 0$,

$$< \alpha^{'}(0), x_0 - y_0 >= 0$$

   So we have $\alpha^{'}(0) \perp l$.
   Then $T_{x_0}S \perp l$ since α is arbitrary.
   Similarly, $T_{y_0}S \perp l$.
   So l cuts S orthogonally at at least two points.

4. (a) Let $\{v_1, v_2\}$ be two unit vectors such that $\{v_1, v_2, a\}$ forms an orthonormal basis of $\mathbb{R}^3$ and,

$$\begin{aligned} v_1 \times v_2 &= a \\ v_2 \times a &= v_1 \\ a \times v_1 &= v_2 \end{aligned}$$

Let $X(\theta, t) = r\cos\theta v_1 + r\sin\theta v_2 + ta$.
Then $X(\theta, t)$ is a patametrization of S.
For any $p_0 \in S$, then $t_0 =< p_0, a >$ and $|p_0|^2 = r^2 + t_0^2$.

$$\begin{aligned} X_\theta &= -r\sin\theta v_1 + r\cos\theta v_2 \\ X_t &= a \\ X_\theta \times X_t &= r\sin\theta v_2 + r\cos\theta v_1 \\ &= p_0 - t_0 a \\ &= p_0 - < p_0, a > a \end{aligned}$$

$v \in T_{p_0}S$
$\Leftrightarrow < v, X_\theta \times X_t >= 0$
$\Leftrightarrow < v, p_0 - < p_0, a > a >= 0$
$\Leftrightarrow < v, p_0 > - < p_0, a >< v, a >= 0$
So $T_pS = \{v \in \mathbb{R}^3 :< v, p > - < p, a >< v, a >= 0\}$

(b) The normal line of S at p is

$$\begin{aligned} \alpha(s) &= p + sX_\theta \times X_t \\ &= p + s(p- < p, a > a) \end{aligned}$$

$\alpha(-1) =< p, a > a$ lies on the axis and $< X_\theta \times X_t, a >=< X_\theta \times a, a >= 0$.
So the normal lines of S cut the axis orthogonally.

(c) Assume S is a connected surface whose normal lines all intersect a fixed straight line $l \subset \mathbb{R}^3$ orthogonally.
Let a be a unit direction vector of l.
Fix $x_0 \in S$, for any $y \in S$, there is a smooth regular curve $\beta(t) : [0, 1] \to S$ such that $\beta(0) = x_0, \beta(1) = y$.
Let $N(p)$ be a unit normal vector field of S.
Then by the assumption, $N//(p- < p, a > a)$.

$$\begin{aligned} 0 &=< \beta'(t), N > \\ &=< \beta'(t), p- < p, a > a > \\ &=< \beta'(t), \beta(t)- < \beta(t), a > a > \\ &=< \beta'(t), \beta(t) > - < \beta'(t), a >< \beta(t), a > \\ &= \frac{1}{2}[< \beta(t), \beta(t) > - < \beta(t), a >^2]' \end{aligned}$$

So $< \beta(t), \beta(t) > - < \beta(t), a >^2= r^2$ for some positive constant r.(r is positive because β contains not only one point)
So $S \subset \{p \in \mathbb{R}^3 : |p|^2- < p, a >^2= r^2\}$ since y is arbitrary.

5. (a) For any $p \in S$, let $X : U \subset \mathbb{R}^2 \to S$ be a parametrization of S around p.
Then $(f \circ X)(u,v) = |X(u,v) - p_0| =< X(u,v) - p_0, X(u,v) - p_0 >^{\frac{1}{2}}$.
By the assumption $p_0 \notin S$, we have $X(u,v) - p_0 \neq 0$ for all $(u,v) \in U$.
So $f \circ X$ is a smooth function near $X^{-1}(p)$.
Then f is a smooth function on S.
Let $\alpha(t) : (-\epsilon, \epsilon) \to S$ be any regular curve on S with $\alpha(0) = p$.

$$\begin{aligned}(f \circ \alpha)^{'}(t) &= [< \alpha(t) - p_0, \alpha(t) - p_0 >^{\frac{1}{2}}]' \\ &= \frac{< \alpha^{'}(t), \alpha(t) - p_0 >}{|\alpha(t) - p_0|}\end{aligned}$$

$(f \circ \alpha)^{'}(0) = 0 \quad \Leftrightarrow \quad < \alpha^{'}(0), p - p_0 >= 0$
So $p \in S$ is a critical point of $f(p)$
$\Leftrightarrow (f \circ \alpha)^{'}(0) = 0$ for any such α
$\Leftrightarrow < \alpha^{'}(0), p - p_0 >= 0$ for any such α
$\Leftrightarrow (p - p_0) \perp T_pS$

(b) Let $Y : U \subset \mathbb{R}^2 \to S$ be a parametrization of S around p.
$(h \circ Y)(u,v) =< Y(u,v), v >$ is a smooth function near $Y^{-1}(p)$.
So h is a smooth function on S.
Let $\beta(t) : (-\epsilon, \epsilon) \to S$ be any regular curve on S with $\beta(0) = p$.

$$\begin{aligned}(h \circ \beta)^{'}(t) &=< \beta(t), v >^{'} \\ &=< \beta^{'}(t), v >\end{aligned}$$

$(h \circ \beta)^{'}(0) = 0 \quad \Leftrightarrow \quad < \beta^{'}(0), v >= 0$
So $p \in S$ is a critical point of $h(p)$
$\Leftrightarrow (h \circ \beta)^{'}(0) = 0$ for any such β
$\Leftrightarrow < \beta^{'}(0), v >= 0$ for any such β
$\Leftrightarrow v \perp T_pS$

6. Let $X : [0, 2\pi) \times [0, 2\pi) \to \mathbb{R}^3$ be

$$X(\theta, \phi) =< \cos\phi(a + r\cos\theta), \sin\phi(a + r\cos\theta), r\sin\theta >$$

Then X is a parametrization of S and X is bijective.
$X_\theta =< -r\cos\phi\sin\theta, -r\sin\phi\sin\theta, r\cos\theta >$
$X_\phi =< -\sin\phi(a + r\cos\theta), \cos\phi(a + r\cos\theta), 0 >$
$|X_\theta|^2 = r^2$, $|X_\phi|^2 = (a + r\cos\theta)^2$, $< X_\theta, X_\phi >= 0$
So the area of S is

$$\begin{aligned}
Area &= \int_0^{2\pi}\int_0^{2\pi} \sqrt{|X_\theta|^2|X_\phi|^2 - < X_\theta, X_\phi >^2} d\phi d\theta \\
&= \int_0^{2\pi}\int_0^{2\pi} r(a + r\cos\theta) d\phi d\theta \\
&= \int_0^{2\pi} (2\pi ar + 2\pi r^2\cos\theta) d\theta \\
&= 4\pi^2 ar
\end{aligned}$$

7. Let a be a regular value of F.
   Then $S = F^{-1}(a)$ is a regular surface and $\nabla F \neq 0$ on S.
   Let $p \in S$, let $\alpha(t) : (-\epsilon, \epsilon) \to S$ be any regular curve on S with $\alpha(0) = p$.

$$\begin{aligned} \alpha(t) &\subset S \\ (F \circ \alpha)(t) &= a \\ (F \circ \alpha)^{'}(0) &= 0 \\ < \nabla F(p), \alpha^{'}(0) > &= 0 \end{aligned}$$

   So $\nabla F(p) \perp T_p S$ since such α is arbitrary.
   F is a smooth function and $\nabla F \neq 0$ on S.
   So $\dfrac{\nabla F}{|\nabla F|}$ defines a smooth unit normal vector field on S.
   Then S is orientable.

8. $X_u =< \cos v, \sin v, 0 >$, $X_v =< -u\sin v, u\cos v, b >$
$g_{uv} = \begin{bmatrix} 1 & 0 \\ 0 & b^2+u^2 \end{bmatrix}$
the area of the surface is

$$\begin{aligned} area &= \int_0^{2\pi}\int_1^3 \sqrt{det(g_{uv})}dudv \\ &= \int_0^{2\pi}\int_1^3 \sqrt{b^2+u^2}dudv \\ &= \int_0^{2\pi} \frac{1}{2}\left[u\sqrt{u^2+b^2} + b^2\ln|u+\sqrt{u^2+b^2}|\right]_1^3 dv \\ &= \pi\left[3\sqrt{9+b^2} - \sqrt{1+b^2} + b^2\ln(3+\sqrt{9+b^2}) - b^2\ln(1+\sqrt{1+b^2})\right] \end{aligned}$$

9. $\psi = \overline{X}^{-1} \circ X$
$\overline{X}(\psi) = X$
$\overline{X}(\overline{u}(u,v), \overline{v}(u,v)) = X(u,v)$
So we have

$$\frac{\partial X}{\partial u} = \frac{\partial \overline{X}}{\partial \overline{u}} \frac{\partial \overline{u}}{\partial u} + \frac{\partial \overline{X}}{\partial \overline{v}} \frac{\partial \overline{v}}{\partial u}$$

$$\frac{\partial X}{\partial v} = \frac{\partial \overline{X}}{\partial \overline{u}} \frac{\partial \overline{u}}{\partial v} + \frac{\partial \overline{X}}{\partial \overline{v}} \frac{\partial \overline{v}}{\partial v}$$

$$\begin{bmatrix} \frac{\partial X}{\partial u} \\ \frac{\partial X}{\partial v} \end{bmatrix} = \begin{bmatrix} \frac{\partial \overline{u}}{\partial u} & \frac{\partial \overline{v}}{\partial u} \\ \frac{\partial \overline{u}}{\partial v} & \frac{\partial \overline{v}}{\partial v} \end{bmatrix} \begin{bmatrix} \frac{\partial \overline{X}}{\partial \overline{u}} \\ \frac{\partial \overline{X}}{\partial \overline{v}} \end{bmatrix}$$

By the assumption

$$\begin{bmatrix} a_1 & a_2 \end{bmatrix} \begin{bmatrix} \frac{\partial X}{\partial u} \\ \frac{\partial X}{\partial v} \end{bmatrix} = \begin{bmatrix} a_1 & a_2 \end{bmatrix} \begin{bmatrix} \frac{\partial \overline{u}}{\partial u} & \frac{\partial \overline{v}}{\partial u} \\ \frac{\partial \overline{u}}{\partial v} & \frac{\partial \overline{v}}{\partial v} \end{bmatrix} \begin{bmatrix} \frac{\partial \overline{X}}{\partial \overline{u}} \\ \frac{\partial \overline{X}}{\partial \overline{v}} \end{bmatrix}$$

$$= \begin{bmatrix} b_1 & b_2 \end{bmatrix} \begin{bmatrix} \frac{\partial \overline{X}}{\partial \overline{u}} \\ \frac{\partial \overline{X}}{\partial \overline{v}} \end{bmatrix}$$

So we get

$$\begin{bmatrix} b_1 & b_2 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 \end{bmatrix} \begin{bmatrix} \frac{\partial \overline{u}}{\partial u} & \frac{\partial \overline{v}}{\partial u} \\ \frac{\partial \overline{u}}{\partial v} & \frac{\partial \overline{v}}{\partial v} \end{bmatrix}$$

which is

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} \frac{\partial \overline{u}}{\partial u} & \frac{\partial \overline{u}}{\partial v} \\ \frac{\partial \overline{v}}{\partial u} & \frac{\partial \overline{v}}{\partial v} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

10. (a)

$$x^2 + y^2 + z^2 = ax$$
$$(x - \frac{a}{2})^2 + y^2 + z^2 = (\frac{a}{2})^2$$

So S_1 is the sphere centered at $(\frac{a}{2}, 0, 0)$ with radius $\frac{a}{2}$.
Similarly, S_2 is the sphere centered at $(0, \frac{b}{2}, 0)$ with radius $\frac{b}{2}$ and S_3 is the sphere centered at $(0, 0, \frac{c}{2})$ with radius $\frac{c}{2}$.

(b) Since S_1 and S_2 are spheres,

$$N_1 = \frac{2}{a}(x - \frac{a}{2}, y, z)$$

$$N_2 = \frac{2}{b}(x, y - \frac{b}{2}, z)$$

are unit normal vector fields of S_1 and S_2.
Let $p = (x_0, y_0, z_0) \in S_1 \cap S_2$, then p satisfies

$$x_0^2 + y_0^2 + z_0^2 = ax_0$$

$$x_0^2 + y_0^2 + z_0^2 = by_0$$

So we have

$$ax_0 = by_0$$

Then we compute the angle between S_1 and S_2,

$$\begin{aligned} < N_1(p), N_2(p) > &= \frac{4}{ab}(x_0^2 - \frac{a}{2}x_0 + y_0^2 - \frac{b}{2}y_0 + z_0^2) \\ &= \frac{4}{ab}(x_0^2 + y_0^2 + z_0^2 - \frac{a}{2}x_0 - \frac{b}{2}y_0) \\ &= \frac{4}{ab}(ax_0 - \frac{a}{2}x_0 - \frac{b}{2}y_0) \\ &= \frac{2}{ab}(ax_0 - by_0) \\ &= 0 \end{aligned}$$

So S_1 meets S_2 orthogonally.
Similarly, S_1, S_2, and S_3 all meet orthogonally.

11. Let $\phi : S_1 \to \mathbb{R}^3$ be

$$\phi(x,y,z) = (\frac{x}{a}, \frac{y}{b}, \frac{z}{c})$$

Then we need to show that ϕ is a bijective differentiable map between S_1 and S_2.

(a) Show that $\phi(S_1) \subset S_2$
Let $(x,y,z) \in S_1$, then

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$
$$(\frac{x}{a})^2 + (\frac{y}{b})^2 + (\frac{z}{c})^2 = 1$$
$$\|\phi(x,y,z)\| = 1$$

So $\phi(x,y,z) \in S_2$.

(b) Show that ϕ is injective

$$\phi(x_1,y_1,z_1) = \phi(x_2,y_2,z_2)$$
$$(\frac{x_1}{a}, \frac{y_1}{b}, \frac{z_1}{c}) = (\frac{x_2}{a}, \frac{y_2}{b}, \frac{z_2}{c})$$
$$(x_1,y_1,z_1) = (x_2,y_2,z_2)$$

So ϕ is injective.

(c) Show that ϕ is surjective
For any $(\overline{x}, \overline{y}, \overline{z}) \in S_2$, $\overline{x}^2 + \overline{y}^2 + \overline{z}^2 = 1$.
Let $(x,y,z) = (a\overline{x}, b\overline{y}, c\overline{z})$, then

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = \overline{x}^2 + \overline{y}^2 + \overline{z}^2 = 1$$

So $(x,y,z) \in S_1$ and $\phi(x,y,z) = (\overline{x}, \overline{y}, \overline{z})$.
This means ϕ is surjective.

(d) Show that ϕ is differentiable
For any $p \in S_1$, let $X : U \to S_1$ be parametrization of S_1 around p.
So $\phi \circ X : \mathbb{R}^2 \to \mathbb{R}^3$ is smooth since X is smooth from $\mathbb{R}^2$ to $\mathbb{R}^3$ and ϕ is smooth from $\mathbb{R}^3$ to $\mathbb{R}^3$.
So ϕ is a differentiable map from S_1 to S_2.