THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH4030 Differential Geometry
Solution of Assignment 2

1. (a) By the definition,

$$\begin{aligned} L_a^b(\alpha) &= \int_a^b |\alpha'(\theta)| d\theta \\ &= \int_a^b \Big| \Big(r'(\theta)\cos\theta - r(\theta)\sin\theta, r'(\theta)\sin\theta + r(\theta)\cos\theta \Big) \Big| d\theta \\ &= \int_a^b \sqrt{r(\theta)^2 + r'(\theta)^2} d\theta \end{aligned}$$

(b) To compute the curvature of α, we need to know α' and α''.

$$\alpha'' = \Big(r''\cos\theta - 2r'\sin\theta - r\cos\theta, r''\sin\theta + 2r'\cos\theta - r\sin\theta \Big)$$

So the curvature is

$$\begin{aligned} \kappa &= \frac{\det(\alpha', \alpha'')}{|\alpha'|^3} \\ &= \frac{2(r')^2 - rr'' + r^2}{[(r')^2 + r^2]^{\frac{3}{2}}} \end{aligned}$$

2. (a) $'' \Rightarrow ''$ Given $\alpha(s)$, a helix which is p.b.a.l. and $\kappa > 0$.
Then we have

$$< \alpha'(s), v_0 >= c_0$$

for a non-zero fixed vector v_0 and a constant c_0.
So we have $< T(s), v_0 >= c_0$.

$$< T(s), v_0 >' = 0$$
$$< \kappa N(s), v_0 >= 0$$
$$< N(s), v_0 >= 0 \quad \text{since } \kappa > 0$$

So we have $< N(s), v_0 >= 0$.
Keep differentiating,

$$< N(s), v_0 >' = 0$$
$$< -\kappa T(s) - \tau B(s), v_0 >= 0$$
$$-\kappa < T(s), v_0 > -\tau < B(s), v_0 >= 0$$
$$-\kappa c_0 - \tau < B(s), v_0 >= 0$$

Now we need to compute $< B(s), v_0 >$,

$$\begin{aligned} < B(s), v_0 >' &=< B'(s), v_0 > \\ &=< \tau N(s), v_0 > \\ &= \tau < N(s), v_0 > \\ &= 0 \end{aligned}$$

So $< B(s), v_0 >= c_1$ for some constant c_1.

$$-\kappa c_0 - \tau < B(s), v_0 >= 0$$
$$-\kappa c_0 - \tau c_1 = 0$$

Now we need to show that $c_1 \neq 0$.
If $c_0 \neq 0$, then $c_1 \neq 0$ since $\kappa > 0$.
If $c_0 = 0$, then

$$\begin{aligned} 0 \neq |v_0|^2 & \\ &=< T(s), v_0 >^2 + < N(s), v_0 >^2 + < B(s), v_0 >^2 \\ &= c_0^2 + c_1^2 \\ &= c_1^2 \end{aligned}$$

So in any case, $c_1 \neq 0$.
Then $\tau = -\dfrac{c_0}{c_1}\kappa$

(b) $'' \Leftarrow''$ Given $\tau = c\kappa$ for some constant c.
Go back to the previous proof, we just let $v(s) = -cT(s) + B(s)$.

$$\begin{aligned} v^{'}(s) &= -cT^{'}(s) + B^{'}(s) \\ &= -c(\kappa N(s)) + \tau N(s) \\ &= (-c\kappa + \tau)N(s) \\ &= (-c\kappa + c\kappa)N(s) \\ &= 0 \end{aligned}$$

So $v(s)$ is a non-zero fixed vector and

$$< \alpha^{'}(s), v >=< T(s), -cT(s) + B(s) >= -c$$

3. We need to show two things, which are α lies on a sphere and α lies on a plane.

   (a) Show that α lies on a sphere.
   By the assumption, $\alpha(s) - x_0 = f(s)N(s)$ for some function $f(s)$.

$$\begin{aligned} <\alpha(s) - x_0, \alpha(s) - x_0 >^{'} &= <\alpha^{'}(s), \alpha(s) - x_0 > \\ &= < T(s), f(s)N(s) > \\ &= 0 \end{aligned}$$

   So $|\alpha(s) - x_0| = R$ for some positive constant R. (If $R = 0$, α will be a single point. A point is not a regular curve).
   Which means α lies on a sphere.

   (b) Show that α lies on a plane.

$$< \alpha(s) - x_0, B(s) > = < f(s)N(s), B(s) > = 0$$

$$< \alpha(s) - x_0, B(s) >^{'} = 0$$

$$< T(s), B(s) > + < f(s)N(s), \tau N(s) > = 0$$

$$f(s)\tau = 0$$

   We get $\tau = 0$ since $f(s)$ is not identically zero.
   So α lies on a plane.

4. (a) $\alpha^{'} = (-a\sin t, b\cos t)$
$\alpha^{''} = (-a\cos t, -b\sin t)$

$$\begin{aligned}\kappa &= \frac{\det(\alpha^{'}, \alpha^{''})}{|\alpha'|^3}\\ &= \frac{ab}{|\alpha'|^3}\\ &> 0\end{aligned}$$

So the ellipse is convex.

(b)

$$\begin{aligned}0 &= \kappa^{'}(s)\\ &= \left(\frac{ab}{|\alpha'|^3}\right)^{'}\\ &= -\frac{3ab(a^2-b^2)}{|\alpha'|^5}\sin t\cos t\\ &= -\frac{3ab(a^2-b^2)}{2|\alpha'|^5}\sin 2t\end{aligned}$$

$$\begin{gathered}\sin 2t = 0 \quad \text{since } a > b > 0\\ t = 0, \frac{\pi}{2}, \pi, \frac{3}{2}\pi, 2\pi\end{gathered}$$

So the ellipse has exactly 4 vertices.

5. (a) Method 1:
   Suppose the cone is a surface.
   Then for $V = \{(x, y, z) \in S| - 1 < z < 1\}$, an open neighborhood of $(0, 0, 0) \in S$, there is an open subset $U \subset R^2$ and a function X such that

$$X : U \to V \quad \text{is a homeomorphism}$$

   Let $p_0 \in U$ with $X(p_0) = (0, 0, 0)$.
   Since U is open, there is a $\epsilon > 0$ such that $\overline{B_\epsilon(p_0)} \subset U$.
   Since X is a homeomorphism, $X(B_\epsilon(p_0))$ is an open neighborhood of $(0, 0, 0)$ in S.
   So $X(B_\epsilon(p_0))\backslash(0, 0, 0))$ is not connected as $X(B_\epsilon(p_0))$ is open in S.
   But $B_\epsilon(p_0)\backslash\{p_0\}$ is connected, this is a contraction since X is a homeomorphism.
   So the cone is not a surface.

   (b) Method 2:
   $f(x, y) = z = \pm\sqrt{x^2 + y^2}$ is not smooth and not one-to-one near (0,0).
   So are $g(y, z) = x = \pm\sqrt{z^2 - y^2}$ and $h(x, z) = y = \pm\sqrt{z^2 - x^2}$.
   So the cone is not a surface

6. (a) S is the graph of $f(x,y) = x^2 - y^2$, so it is a surface.

   (b)
$$(u+v)^2 - (u-v)^2 = 4uv$$
So $X_1(u,v) \in S$.
$$\frac{\partial X_1}{\partial u} = (1, 1, 4v)$$
$$\frac{\partial X_1}{\partial v} = (1, -1, 4u)$$
So $\{\frac{\partial X_1}{\partial u}, \frac{\partial X_1}{\partial v}\}$ is linearly independent.
Then $X_1(u,v)$ is a parametrization.

   (c)
$$(u \cosh v)^2 + (u \sinh v)^2 = u^2$$
So $X_2(u,v) \in S$.
$$\frac{\partial X_2}{\partial u} = (\cosh v, \sinh v, 2u)$$
$$\frac{\partial X_2}{\partial v} = (u \sinh v, u \cosh v, 0)$$
So $\{\frac{\partial X_2}{\partial u}, \frac{\partial X_2}{\partial v}\}$ is linearly independent as $u \neq 0$.
Then $X_2(u,v)$ is a parametrization.

7. (a) $dF = (0, 0, 2z)$
   When $F = z^2 = 0$, $z = 0$.
   $dF(0, 0, 0) = (0, 0, 0)$
   So 0 is not a regular value of F.

   (b) $F^{-1}(0) = \{(x, y, 0)|(x, y) \in \mathbb{R}^2\}$
   So $F^{-1}(0)$ is the xOy plane, and it is a surface.