Tutorial 7: Conformal map.

① Def: A diffeomorphism $\varphi: S \to \bar{S}$ is called a conformal map if for all $p \in S$, and all $v_1, v_2 \in T_pS$ we have

$$\langle d\varphi_p(v_1), d\varphi_p(v_2)\rangle = \lambda^2(p)\langle v_1, v_2\rangle_p$$

where λ^2 is a nowhere-zero differentiable function on S, the surfaces S and $\bar{S}$ are then said to be conformal.

② Prop: Let $\varphi: S \to \bar{S}$ be a conformal map, then it is angle-preserving. i.e, let $\alpha: (-\varepsilon, \varepsilon) \to S$, $\beta: (-\varepsilon, \varepsilon) \to S$ be two regular curves on S with $\alpha(0) = \beta(0) = p$, then

$$\frac{\langle \alpha'(0), \beta'(0)\rangle}{|\alpha'(0)|\cdot|\beta'(0)|} = \frac{\langle d\varphi_p(\alpha'(0)), d\varphi_p(\beta'(0))\rangle}{|d\varphi(\alpha'(0))|\cdot|d\varphi(\beta'(0))|}$$

Pf: By the definition of conformal map,

$$\frac{\langle d\varphi(\alpha'(0)), d\varphi(\beta'(0))\rangle}{|d\varphi(\alpha'(0))|\cdot|d\varphi(\beta'(0))|} = \frac{\lambda^2(p)\langle \alpha'(0), \beta'(0)\rangle}{|\lambda\alpha'(0)|\cdot|\lambda\beta'(0)|} = \frac{\langle \alpha'(0), \beta'(0)\rangle}{|\alpha'(0)|\cdot|\beta'(0)|}$$

③ Thm: Any two regular surfaces are locally conformal

- the proof is based on the existence of isothermal coordinates, i.e. let S be any regular surface, $\forall p \in S$, we can find a parametrization of S around p such that

$$X: U \subseteq \mathbb{R}^2 \to S$$

with $$g_{ij} = \begin{bmatrix} \langle X_u, X_u\rangle & \langle X_u, X_v\rangle \\ \langle X_u, X_v\rangle & \langle X_v, X_v\rangle \end{bmatrix} = \lambda^2(u,v)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

④ Example: Stereographic projection.

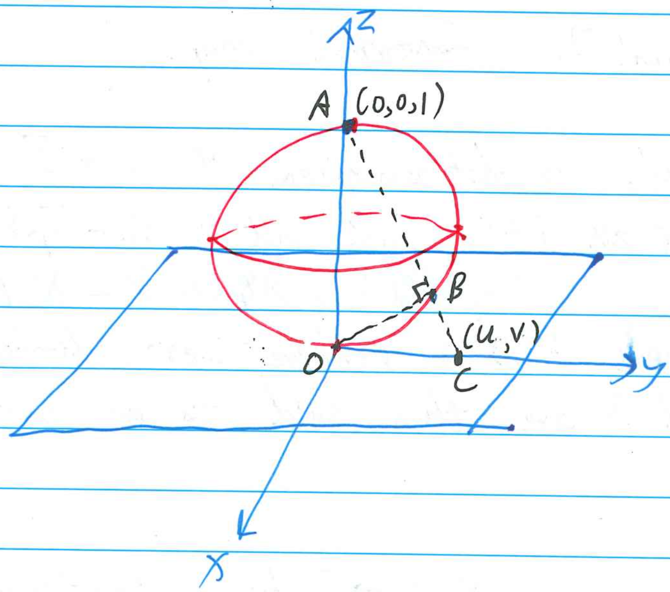
let S be the sphere centered at $(0,0,\frac{1}{2})$ with radius $\frac{1}{2}$.

$A=(0,0,1)$

for $\forall\ B \in S \setminus \{A\}$

the straight line AB will intersect the xoy plane at a unique point C.

So there is a bijective map between xy plane and $S\setminus\{A\}$, we call this stereographic projection.



[Compute the formula for this map].

let $C=(u,v,0)$

then $OB \perp AC$

$$\therefore \frac{|CB|}{|CO|} = \frac{|CO|}{|AC|}$$

$$\therefore |CB| = \frac{\sqrt{u^2+v^2}\cdot\sqrt{u^2+v^2}}{\sqrt{u^2+v^2+1}} = \frac{u^2+v^2}{\sqrt{u^2+v^2+1}}$$

$$\therefore \frac{|CB|}{|CA|} = \frac{u^2+v^2}{u^2+v^2+1}$$

$$\therefore \vec{CB} = \frac{|CB|}{|CA|}\cdot\vec{CA} = \frac{u^2+v^2}{u^2+v^2+1}(-u,-v,1)$$

$$\because \vec{OB} = \vec{OC} + \vec{CB}$$

$$= (u,v,0) + \frac{u^2+v^2}{u^2+v^2+1}(-u,-v,1)$$

$$= \left(\frac{u}{u^2+v^2+1}, \frac{v}{u^2+v^2+1}, \frac{u^2+v^2}{u^2+v^2+1}\right)$$

- $X: R^2 \to S \setminus \{A\}$ given by
$$X(u,v) = \left(\frac{u}{u^2+v^2+1}, \frac{v}{u^2+v^2+1}, \frac{u^2+v^2}{u^2+v^2+1} \right)$$
is a bijective diffeomorphism.

- Check X is a conformal map.
$$X_u = \left(\frac{1+v^2-u^2}{[1+u^2+v^2]^2}, \frac{-2uv}{[1+u^2+v^2]^2}, \frac{2u}{[1+u^2+v^2]^2} \right)$$
$$X_v = \left(\frac{-2uv}{[1+u^2+v^2]^2}, \frac{1+u^2-v^2}{[1+u^2+v^2]^2}, \frac{2v}{[1+u^2+v^2]^2} \right)$$
$\therefore$
$$g = \begin{bmatrix} \frac{1}{[1+u^2+v^2]^2} & 0 \\ 0 & \frac{1}{[1+u^2+v^2]^2} \end{bmatrix}$$
$$= \frac{1}{[1+u^2+v^2]^2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$\therefore$ X is an isothermal coordinate, then it is a conformal map between R^2 and $S \setminus \{A\}$.