Tutorial 6: Review

Curve:

① Plane curve:

• Let $\alpha(s): I \to R^2$ be a plane curve p.b.a.l.

$$\begin{cases} \text{Frenet frame: } \{\alpha'(s), J(\alpha'(s))\} = \{T(s), N(s)\} \\ k(s) = \langle T'(s), N(s) \rangle \end{cases}$$

• When $\alpha(t)$ is not p.b.a.l., we can also define the Frenet frame and the signed curvature by reparametrization.

Let $t = \phi(s)$ be an increasing function such that $\alpha(\phi(s))$ is p.b.a.l.

$\therefore \quad 1 = |\alpha'(t) \cdot \phi'(s)|$

$\therefore \quad \phi'(s) = \dfrac{1}{|\alpha'(t)|}$

$T(s) = \dfrac{\partial}{\partial s}\alpha(\phi(s)) = \alpha'(t) \cdot \phi'(s) = \alpha'(t) \cdot \dfrac{1}{|\alpha'(t)|}$

$$\begin{cases} \text{Frenet frame: } \left\{ \dfrac{\alpha'(t)}{|\alpha'(t)|}, J\left(\dfrac{\alpha'(t)}{|\alpha'(t)|}\right) \right\} \\ k(t) = \dfrac{\det(\alpha'(t), \alpha''(t))}{|\alpha'(t)|^3} \end{cases}$$

② Space curve:

• Let $\alpha(s): I \to R^3$ be a space curve p.b.a.l.

$$\begin{cases} \text{Frenet frame: } \{T, N, B\} = \left\{\alpha'(s), \dfrac{\alpha''(s)}{|\alpha''(s)|}, T \times N \right\} \\ k = |T'(s)| \\ \tau = \langle B'(s), N(s) \rangle \end{cases}$$

• When $\alpha(t)$ is not p.b.a.l. we can also define the Frenet frame, k and τ by reparametrization.

Let $t = \phi(s)$ be an increasing function such that $\alpha(\phi(s))$ is p.b.a.l.

$\therefore \quad 1 = |\alpha'(t) \cdot \phi'(s)|$

$\phi'(s) = \dfrac{1}{|\alpha'(t)|}$

$T(s) = \dfrac{\partial}{\partial s}\alpha(\phi(s)) = \alpha'(t) \cdot \phi'(s) = \dfrac{\alpha'(t)}{|\alpha'(t)|}$

$N(s) = \dfrac{T'(s)}{|T'(s)|} = \dfrac{\frac{\partial}{\partial t}\left(\frac{\alpha'(t)}{|\alpha'(t)|}\right) \cdot \phi'(s)}{\left|\frac{\partial}{\partial t}\left(\frac{\alpha'(t)}{|\alpha'(t)|}\right) \cdot \phi'(s)\right|} = \dfrac{T'(t)}{|T'(t)|}$

$B(s) = T(s) \times N(s)$

Frenet frame: $T(t) = \dfrac{\alpha'(t)}{|\alpha'(t)|}$, $N(t) = \dfrac{T'(t)}{|T'(t)|}$, $B(t) = T(t) \times N(t)$

$$\begin{cases} k = \dfrac{|\alpha'(t) \times \alpha''(t)|}{|\alpha'|^3} \\ \tau = -\dfrac{\langle \alpha' \times \alpha'', \alpha''' \rangle}{|\alpha' \times \alpha''|^2} \end{cases}$$

• **Remark:** if $\alpha(t)$ is not p.b.a.l.

$$k(s) = |T'(s)| = |T'(t) \cdot \phi'(s)| \neq |T'(t)|$$

$$\tau(s) = \langle B'(s), N(s) \rangle = \langle B'(t) \cdot \phi'(s), N(t) \rangle \neq \langle B'(t), N(t) \rangle$$

if we want to compute k and τ by definition, we must differentiate everything in terms of the arc-length parameter s.

③ Example:

Let $\alpha(t): (1,2) \to R^3$ be $\alpha(t) = (\frac{1}{2}t^2, -t\cos t + \sin t, t\sin t + \cos t)$

Find the Frenet frame and k, τ.

Ans: $\alpha'(t) = (t, t\sin t, t\cos t)$

$\therefore |\alpha'(t)| = \sqrt{2}\, t$

$$\therefore \begin{cases} T(t) = \dfrac{\alpha'(t)}{|\alpha'(t)|} = \dfrac{1}{\sqrt{2}}(1, \sin t, \cos t) \\ N(t) = \dfrac{T'(t)}{|T'(t)|} = (0, \cos t, -\sin t) \\ B(t) = T \times N = \left(-\dfrac{1}{\sqrt{2}}, \dfrac{\sin t}{\sqrt{2}}, \dfrac{\cos t}{\sqrt{2}}\right) \end{cases}$$

[method 1 to compute k, τ]

$\alpha'(t) = (t, t\sin t, t\cos t)$

$\alpha''(t) = (1, \sin t + t\cos t, \cos t - t\sin t)$

$\alpha'''(t) = (0, 2\cos t - t\sin t, -2\sin t - t\cos t)$

$\alpha' \times \alpha'' = (-t^2, t^2\sin t, t^2\cos t) = t^2(-1, \sin t, \cos t)$

$$\therefore k = \frac{|\alpha' \times \alpha''|}{|\alpha'|^3} = \frac{\sqrt{2}\,t^2}{2\sqrt{2}\,t^3} = \frac{1}{2t}$$

$$\tau = -\frac{\langle \alpha' \times \alpha'', \alpha''' \rangle}{|\alpha' \times \alpha''|^2} = -\frac{-t^3}{2t^4} = \frac{1}{2t}$$

[Method 2 to compute k, τ]

let $t = t(s)$ be an increasing function such that $\alpha(t(s))$ is p.b.a.l.

then $1 = \left|\frac{\partial}{\partial s}(\alpha(t(s)))\right| = |\alpha'(t)| \cdot t'(s)$

$$\therefore \quad t'(s) = \frac{1}{|\alpha'(t)|} = \frac{1}{\sqrt{2}t}$$

$$T(s) = \frac{\partial}{\partial s}(\alpha(t(s))) = \frac{\alpha'(t)}{|\alpha'(t)|} = T(t)$$

$$N(s) = \frac{T'(s)}{|T'(s)|} = \frac{T'(t) \cdot t'(s)}{|T'(t)| \cdot t'(s)} = \frac{T'(t)}{|T'(t)|} = N(t)$$

$$B(s) = T(s) \times N(s) = T(t) \times N(t)$$

$$\therefore \quad k = |T'(s)| = |T'(t) \cdot t'(s)| = \frac{\left|\left(0, \frac{\cos t}{\sqrt{2}}, -\frac{\sin t}{\sqrt{2}}\right)\right|}{\sqrt{2}t} = \frac{1}{2t}$$

$$\tau = \langle B'(s), N(s) \rangle = \langle B'(t) \cdot t'(s), N(t) \rangle$$

$$= \left\langle \frac{1}{\sqrt{2}}(0, \cos t, -\sin t) \cdot \frac{1}{\sqrt{2}t}, (0, \cos t, -\sin t) \right\rangle$$

$$= \frac{1}{2t}$$

- since $\alpha(t)$ is not p.b.a.l

$$\therefore \quad k \neq |T'(t)|$$

$$\tau \neq \langle B'(t), N(t) \rangle$$

i.e. the form of the Frenet frame $\{T(t), N(t), B(t)\}$ is still the same, but to compute k and τ, we need to differentiate everything in terms of s, not t.

Surface:

① we have three ways to check if $S \subseteq \mathbb{R}^3$ is a regular surface.

1°. if $\forall p \in S$, $\exists$ a nbd $V \subseteq S$ of p and a smooth map

$$X: U \subseteq \mathbb{R}^2 \to V$$

s.t. $\begin{cases} X \text{ is a homeomorphism} \\ dX \text{ is of full rank.} \end{cases}$ (X_1, X_2 are linearly independent)

2°. if $\forall p \in S$, $\exists$ a nbd $V \subseteq S$ of p such that V is a smooth graph on xy or yz or xz plane

3°. if $S = F^{-1}(a)$ where F is a smooth function on $\mathbb{R}^3$ and a is a regular value of F.

② Example:

Show that the sphere is a regular surface

Pf: By the symmetry of S, we only need to show S is a regular surface near $(0,0,1)$

near $(0,0,1)$, S is the graph of $z = \sqrt{1-x^2-y^2}$

$\therefore S$ is a regular surface near $(0,0,1)$.

③ Example:

Show that $S = \{(x,y,z) : x^2+y^2=z^2,\ z \geq 0\}$ is not a regular surface.

Pf: the problem happens at $(0,0,0)$.

near $(0,0,0)$, $x = \pm\sqrt{z^2-y^2}$ is not a function

$y = \pm\sqrt{z^2-x^2}$ is not a function

$z = \sqrt{x^2+y^2}$ is a function, but it is not differentiable.

$\therefore S$ is not a regular surface near $(0,0,0)$

④ Differentiable functions on surfaces.

Let S_1, S_2 be two regular surfaces and $f: S_1 \to S_2$ is a continuous map.

Then we say that f is differentiable on S_1 if $\forall p \in S_1$, $\exists$

$X: U \subseteq \mathbb{R}^2 \to S_1$, a parametrization of S_1 near p

such that $f \circ X$ is a differentiable map from $U \subseteq \mathbb{R}^2$ to $\mathbb{R}^3$.

⑤ Example: let $S_1 = \{(x,y,z) : x^2+y^2+z^2=1\}$

$S_2 = \{(x,y,z) : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1\}$, $a,b,c>0$

let $f(x,y,z) = (ax, by, cz)$

Show that f is a differentiable map from S_1 to S_2.

Pf: Obviously, $f(S_1) \subseteq S_2$

$\forall p \in S_1$, let $X: U \subseteq \mathbb{R}^2 \to S_1$ be a parametrization of S_1 near p.

then $X(u,v) = (X_1(u,v), X_2(u,v), X_3(u,v))$

where X_1, X_2, X_3 are smooth functions

$f \circ X(u,v) = (aX_1(u,v), bX_2(u,v), cX_3(u,v))$

is smooth since X_1, X_2, X_3 are smooth

$\therefore f: S_1 \to S_2$ is smooth.

⑥ Area, first fundamental form, second fundamental form, curvature.

Let $X(x_1, x_2) : U \to S$ be a parametrization of S

then
- $\text{Area} = \int_U \sqrt{\det g_{ij}}\, d\mathbb{R}^2$
- $g_{ij} = \begin{bmatrix} \langle X_1, X_1 \rangle & \langle X_1, X_2 \rangle \\ \langle X_1, X_2 \rangle & \langle X_2, X_2 \rangle \end{bmatrix}$
- $N = \frac{X_1 \times X_2}{|X_1 \times X_2|}$
- $A_{ij} = \begin{bmatrix} \langle X_{11}, N \rangle & \langle X_{12}, N \rangle \\ \langle X_{12}, N \rangle & \langle X_{22}, N \rangle \end{bmatrix}$

from the definition, A, S, H depend on the direction of N.

- $S = g_{ij}^{-1} A_{ij}$
- $K = \frac{\det(A)}{\det(g)}$, $H = tr(g_{ij}^{-1} A_{ij})$

⑦ Example: Let $S = \{(x,y,z) \in R^3 : z = x^2+y^2,\ x^2+y^2 \le 4\}$

Compute K, H with respect to the "upward" pointing unit normal N, and the area of S.

Ans: 1°. Compute g_{ij}

$$X(x,y) = (x, y, x^2+y^2)$$
$$X_x = (1, 0, 2x)$$
$$X_y = (0, 1, 2y)$$
$$\therefore g_{ij} = \begin{bmatrix} 1+4x^2 & 4xy \\ 4xy & 1+4y^2 \end{bmatrix}$$

2°. Compute N and A_{ij}

$$N = \frac{X_x \times X_y}{|X_x \times X_y|} = \frac{(-2x, -2y, 1)}{\sqrt{1+4x^2+4y^2}}$$
$$X_{xx} = (0,0,2)$$
$$X_{xy} = (0,0,0)$$
$$X_{yy} = (0,0,2)$$
$$A_{ij} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \cdot \frac{1}{\sqrt{1+4x^2+4y^2}}$$

3°. Compute S, K, H

$$S = (g_{ij})^{-1} A_{ij} = \frac{1}{1+4x^2+4y^2} \begin{bmatrix} 1+4y^2 & -4xy \\ -4xy & 1+4x^2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \frac{2}{\sqrt{1+4x^2+4y^2}}$$
$$= \frac{2}{[1+4x^2+4y^2]^{\frac{3}{2}}} \begin{bmatrix} 1+4y^2 & -4xy \\ -4xy & 1+4x^2 \end{bmatrix}$$
$$K = \frac{\det(A)}{\det(g)} = \det(S) = \frac{4}{[1+4x^2+4y^2]^2}$$
$$H = tr(S) = tr((g_{ij})^{-1} A_{ij}) = \frac{4(1+2x^2+2y^2)}{[1+4x^2+4y^2]^{\frac{3}{2}}}$$

4°. Area $= \int \sqrt{1+4x^2+4y^2}\, dR^2 = \int_0^{2\pi} \int_0^2 \sqrt{1+4r^2}\, r\,dr\,d\theta$
$$= \int_0^{2\pi} \left[\frac{1}{12}(1+4r^2)^{\frac{3}{2}}\right]_0^2 d\theta = \int_0^{2\pi} \frac{17^{\frac{3}{2}}-1}{12} d\theta$$
$$= \frac{\pi}{6}(17^{\frac{3}{2}}-1)$$