Tutorial 5.

① Def: The angle of two intersecting surfaces at an intersecting point P is the angle of their tangent planes (or their normal line), i.e.,

$$\cos\theta = \left|\frac{\langle N_1(P), N_2(P)\rangle}{|N_1(P)|\cdot|N_2(P)|}\right| = |\langle N_1(P), N_2(P)\rangle|$$

② Show that $S_1=\{(x,y,z): x^2+y^2+z^2=ax\}$, $S_2=\{(x,y,z): x^2+y^2+z^2=by\}$ intersect orthogonally. $(a,b\neq 0)$

pf: $\because$ $x^2+y^2+z^2=ax$

$\therefore$ $(x-\frac{a}{2})^2+y^2+z^2=\frac{a^2}{4}$

$\therefore$ S_1 is the sphere centered at $(\frac{a}{2},0,0)$ with radius $\frac{a}{2}$

Similarly, S_2 is the sphere centered at $(0,\frac{b}{2},0)$ with radius $\frac{b}{2}$

$$\therefore \begin{cases} N_1=\frac{2}{a}(x-\frac{a}{2}, y, z) \\ N_2=\frac{2}{b}(x, y-\frac{b}{2}, z) \end{cases}$$

at the intersection points,

$$\begin{cases} x^2+y^2+z^2=ax \\ x^2+y^2+z^2=by \end{cases} \Rightarrow ax=by$$

$\therefore$ if (x,y,z) is an intersection point,

$$\langle N_1, N_2\rangle = \frac{4}{ab}(x^2-\frac{a}{2}x+y^2-\frac{b}{2}y+z^2)$$

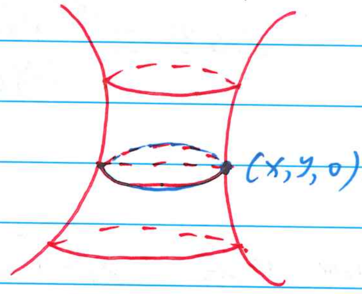
$$=\frac{4}{ab}(ax-\frac{a}{2}x-\frac{b}{2}y)=\frac{4}{ab}(\frac{a}{2}x-\frac{b}{2}y)=\frac{2}{ab}(ax-by)=0$$

$\therefore$ S_1 and S_2 intersect orthogonally.

③ Determine the tangent plans of $x^2+y^2-z^2=1$ at the points $(x, y, 0)$ and show that they are all parallel to the z-axis.

Method1: (a) let $p=(x_0, y_0, 0)\in S$

then $x_0^2+y_0^2=1$



- $\because p\in\alpha(t)=\{(\cos t, \sin t, 0)\}\subseteq S$

$\therefore \alpha'(p)\in T_p(S)$

$\therefore (-y_0, x_0, 0)\in T_p(S)$

- $\because p\in\beta(t)=\{(x_0\cosh t, y_0\cosh t, \sinh t)\}\subseteq S$

$\therefore \beta'(p)\in T_pS$

$\therefore (0,0,1)\in T_p(S)$

$\because (-y_0, x_0, 0), (0,0,1)\in T_pS$ and they are linearly independent

$\therefore T_p(S)=\text{span}\{(-y_0, x_0, 0), (0,0,1)\}$

(b) $\because (0,0,1)\in T_p(S)$

$\therefore$ z-axis is parallel to $T_p(S)$

Method2: $X: R^2\to R^3$ given by

$X(\theta, t)=(\cosh t\cdot\cos\theta, \cosh t\cdot\sin\theta, \sinh t)$

is a parameterization of S

$\therefore X_\theta=(-\cosh t\cdot\sin\theta, \cosh t\cdot\cos\theta, 0)$

$X_t=(\sinh t\cdot\cos\theta, \sinh t\cdot\sin\theta, \cosh t)$

let $p=(x_0, y_0, 0)\in S$

$\therefore \sinh t=0 \Rightarrow t=0$

$\therefore X_\theta(p)=(-1\cdot\sin\theta, 1\cdot\cos\theta, 0)=(-y_0, x_0, 0)$

$X_t(p)=(0,0,1)$

$\therefore T_p(S)=\text{span}\{(-y_0, x_0, 0), (0,0,1)\}$

Remark: if it is difficult to find a parameterization of S, we can use method1 to compute the tangent plane of S.

4. (chain rule) Show that if $\varphi: S_1 \to S_2$ and $\psi: S_2 \to S_3$ are differentiable maps and $p \in S_1$, then

$$d(\psi \circ \varphi)_p = d\psi_{(\varphi(p))} \circ d\varphi_p .$$

pf: $\forall\, p \in S_1$, $\forall\, \vec{v} \in T_pS_1$

there exist

1° $\alpha(t): (-\varepsilon, \varepsilon) \to S_1$ with

$\alpha(0) = p$, $\alpha'(0) = \vec{v}$

2° $X: U \to S_1$ such that

X is a parametrization of S_1 near p and $U \subseteq R^2$

3° $Y: V \to S_2$ such that

Y is a parametrization of S_2 near $\varphi(p)$ and $V \subseteq R^2$

$\therefore$ By the definition

$$d(\psi \circ \varphi)_p(\vec{v})$$

$$= \left.\frac{d}{dt}\right|_{t=0} (\psi \circ \varphi \circ \alpha(t))$$

$$= \left.\frac{d}{dt}\right|_{t=0} \left[(\psi \circ Y) \circ (Y^{-1} \circ \varphi \circ X) \circ (X^{-1} \circ \alpha(t))\right]$$

$$= d(\psi \circ Y)_{Y^{-1}(\varphi(p))} \circ d(Y^{-1} \circ \varphi \circ X)_{X^{-1}(p)} \circ d(X^{-1} \circ \alpha)(0)$$

since $X^{-1} \circ \alpha: R \to R^2$
$Y^{-1} \circ \varphi \circ X: R^2 \to R^2$
$\psi \circ Y: R^2 \to S_3$

- $d(\varphi)(\vec{v})$

$$= \left.\frac{d}{dt}\right|_{t=0} (Y \circ Y^{-1} \circ \varphi \circ X \circ X^{-1} \circ \alpha)$$

$$= dY_{Y^{-1}(\varphi(p))} \circ d(Y^{-1} \circ \varphi \circ X)_{X^{-1}(p)} \circ d(X^{-1} \circ \alpha)(0)$$

since $X^{-1} \circ \alpha: R \to R^2$
$Y^{-1} \circ \varphi \circ X: R^2 \to R^2$
$Y: R^2 \to S_3$

$d\psi$

$$= d(\psi \circ Y \circ Y^{-1})$$

$$= d(\psi \circ Y) \circ d(Y^{-1})$$

since $Y^{-1}: S_2 \to R^2$
$\psi \circ Y: R^2 \to S_3$

$\therefore\ d(\psi \circ \varphi) = d\psi_{(\varphi(p))} \circ d\varphi_p$