Tutorial 3. Convex curves and The Four-Vertex Theorem.

① Def: A regular plane curve $\alpha(s)$ is convex if for all s_0, $\alpha(s)$ lies on one side of the tangent line of $\alpha(s)$ at $s=s_0$.

② Let $\alpha(s): [a,b] \to R^2$ be a simple closed regular curve.
The the followings are equivalent

(a) α is convex

(b) $k(s) \geq 0$ for all s
or $k(s) \leq 0$ for all s

(c) $\forall p, q \in \alpha$, $p \neq q$
p, q divide α into two parts β and γ
then β lies on one side of l_{pq} (where l_{pq} is the straight line joining p and q)
and γ lies on one side of l_{pq}
and β, γ do not lie on the same side of l_{pq}
(i.e. there are no $A \in \beta$, $B \in \gamma$ with A, B lie on the same side of $R^2 \setminus l_{pq}$)

③ lemma: let $\alpha: [0, l] \to R^2$ be a plane closed curve p.b.a.l and let A, B, C be any real numbers, then

$$\int_0^l (Ax + By + C) \frac{dk}{ds} \cdot ds = 0$$ where $\alpha(s) = (x(s), y(s))$

pf: $\because$ $1 = |\alpha'(s)| = \sqrt{(x'(s))^2 + (y'(s))^2}$

$\therefore$ there is a differentiable function $\theta(s): [0, l] \to R$ such that

$$\alpha'(s) = (x'(s), y'(s)) = (\cos\theta(s), \sin\theta(s))$$

$\therefore$ $\alpha''(s) = \langle -\sin\theta, \cos\theta \rangle \cdot \theta'(s)$

$k = \langle \alpha'', J(\alpha') \rangle = \langle (-\sin\theta, \cos\theta) \cdot \theta', (-\sin\theta, \cos\theta) \rangle = \theta'$

$$\therefore \begin{cases} x'' = -\sin\theta \cdot \theta' = -ky' \\ y'' = \cos\theta \cdot \theta' = kx' \end{cases}$$

$$\therefore \int_0^l k' ds = k(l) - k(0) = 0$$

$$\int_0^l x \cdot k' ds = -\int_0^l k \cdot x' ds = -\int_0^l y'' ds = -(y'(l) - y'(0)) = 0$$

$$\int_0^l y k' ds = -\int_0^l k \cdot y' ds = \int_0^l x'' ds = x'(l) - x'(0) = 0$$

④ [Four-Vertex Theorem]

A simple closed convex curve has at least four vertexs. ($k'(s)=0$)

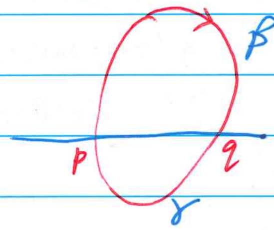
pf: $\because$ $k(s)$ is continuous

$\therefore$ $\exists$ s_1, s_2 such that

$k(s_1)$ is a maximum, $k(s_2)$ is a minimum

let $\alpha(s_1)=p$, $\alpha(s_2)=q$

We can rotate the curve such that $L_{pq}=\{y=0\}$



$\because$ α is convex

$\therefore$ p, q divides α into two parts β, γ

if $k'(s)$ has no other roots

then $\begin{cases} k'>0 \text{ on } \beta,\ k'<0 \text{ on } \gamma & — (*) \\ \text{or } k'<0 \text{ on } \beta,\ k'>0 \text{ on } \gamma & — (**) \end{cases}$

$\because$ $0=\int_\alpha y\cdot k'ds=\int_\beta y\cdot k'ds+\int_\gamma y\cdot k'ds$

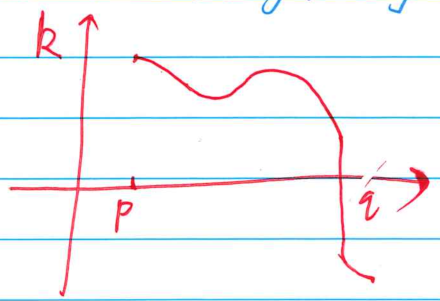
>0 if $(*)$ happens

<0 if $(**)$ happens

Both give us a contradiction

$\therefore$ $k'(s)$ has a third root. and $k'(s)$ changes sign on β or γ

Assume $k'(s)$ changes sign on β



$\because$ p is a maximum, q is a minimum, $k'(s)=0$ for some s between s_1 and s_2, $k(s)$ is not monotone on (s_1, s_2)

$\therefore$ there are at least four vertexs.