Tutorial 2 :

1. Let $\alpha(t): I \to R^3$ be a regular space curve.
Let $\beta(s) = \alpha(\phi(s))$ be p.b.a.l. where $\phi(s)$ is an increasing function.
(a) show that

$$\phi'(s) = \frac{1}{|\alpha'(t)|}, \quad \phi''(s) = \frac{-\langle \alpha', \alpha'' \rangle}{|\alpha'|^4}$$

(b) show the curvature of α is

$$k = \frac{|\alpha' \times \alpha''|}{|\alpha'|^3}$$

Ans: (a) $\because \ |\beta'(s)| \equiv 1$

$\therefore$ By the chain rule,

$$|\phi'(s)| \cdot |\alpha'(t)| = 1$$

$$\therefore \ \phi'(s) = \frac{1}{|\alpha'(t)|}$$

Differentiate again,

$$\phi''(s) = \left[\langle \alpha'(\phi(s)), \alpha'(\phi(s)) \rangle^{-\frac{1}{2}} \right]'$$

$$= -\frac{1}{2} \langle \alpha'(\phi(s)), \alpha'(\phi(s)) \rangle^{-\frac{3}{2}} \cdot 2 \langle \alpha', \alpha'' \cdot \phi' \rangle$$

$$= -\frac{\phi'}{|\alpha'|^3} \langle \alpha', \alpha'' \rangle = -\frac{1}{|\alpha'|^4} \langle \alpha', \alpha'' \rangle$$

(b) $T = \beta'(s) = \alpha'(\phi(s)) \cdot \phi'(s)$

$T' = \alpha''(\phi')^2 + \alpha' \phi''$

$$k^2 = |T'|^2 = (\phi')^4 |\alpha''|^2 + 2(\phi')^2 \phi'' \langle \alpha', \alpha'' \rangle + (\phi'')^2 |\alpha'|^2$$

$$= \frac{|\alpha''|^2}{|\alpha'|^4} - 2 \cdot \frac{1}{|\alpha'|^2} \cdot \frac{\langle \alpha', \alpha'' \rangle^2}{|\alpha'|^4} + \frac{\langle \alpha', \alpha'' \rangle^2}{|\alpha'|^8} |\alpha'|^2$$

$$= \frac{|\alpha''|^2}{|\alpha'|^4} - \frac{\langle \alpha', \alpha'' \rangle^2}{|\alpha'|^6} = \frac{1}{|\alpha'|^6} \left[|\alpha'|^2 |\alpha''|^2 - |\alpha'|^2 |\alpha''|^2 \cos^2\theta \right]$$

$$= \frac{1}{|\alpha'|^6} |\alpha'|^2 |\alpha''|^2 \sin^2\theta = \frac{|\alpha' \times \alpha''|^2}{|\alpha'|^6}$$ where θ is the angle between α' and α''.

$$\therefore \ k = \frac{|\alpha' \times \alpha''|}{|\alpha'|^3}$$

2, Example: Let $\alpha(t) = (t^2, \sin t, \cos t)$

Compute the curvature of α.

Ans: $\alpha'(t) = (2t, \cos t, -\sin t)$

$\alpha''(t) = (2, -\sin t, -\cos t)$

$\alpha' \times \alpha'' = (-1, 2t\cos t - 2\sin t, -2t\sin t - 2\cos t)$

$|\alpha' \times \alpha''| = \sqrt{1 + 4t^2 + 4} = \sqrt{4t^2+5}$

$\therefore k = \dfrac{|\alpha' \times \alpha''|}{|\alpha'|^3} = \dfrac{\sqrt{4t^2+5}}{(4t^2+1)^{\frac{3}{2}}}$

3, Show that a regular space curve is planar if and only if its torsion is identically zero.

Pf: "$\Rightarrow$" W.L.O.G, we may assume $\alpha(s) \subseteq \mathbb{R}^2 \times \{0\}$

$\therefore T = (x'(s), y'(s), 0)$

$N = \dfrac{T'}{|T'|} = \dfrac{(x''(s), y''(s), 0)}{|T'|}$

$\therefore B(s) = (0,0,1)$ or $(0,0,-1)$

$\because B(s)$ is continuous

$\therefore B(s) \equiv (0,0,1)$ for all s

or $B(s) \equiv (0,0,-1)$ for all s

$\therefore B'(s) = 0$

$\therefore \tau = \langle B', N\rangle \equiv 0$

"$\Leftarrow$" $0 = \tau = \langle B', N\rangle$

$0 = \langle B, B\rangle' = 2\langle B', B\rangle$

$\because \langle B', T\rangle = -\langle B, T'\rangle = -\langle B, kN\rangle = 0$

$\because \{T, B, N\}$ is a basis

$\therefore B' = 0$ $\therefore B = \vec{n}$ for some fixed unit vector $\vec{n}$

Let $f(s) = \langle \alpha(s) - \alpha(s_0), \vec{n}\rangle$

$\therefore \begin{cases} f(s_0) = 0 \\ f'(s) = \langle \alpha'(s), \vec{n}\rangle = \langle T, B\rangle = 0 \end{cases}$

$\therefore f(s) \equiv 0$

$\therefore \alpha(s)$ lies on some plane.