Tutorial 1: Signed curvature of a plane curve.

① Let $\alpha(s): I \to R^2$ be a curve p.b.a.l.
then we define the signed curvature of $\alpha(s)$ to be

$$k(s) = \langle \alpha''(s), J(\alpha'(s)) \rangle$$

② [what if $\alpha(t)$ is not parametrized by arc length?]
let $\alpha(t): I \to R^2$ be a regular curve.

1°, $\exists\ \phi(s): I' \to I$ such that

$$\begin{cases} t = \phi(s) \text{ is increasing} \\ \alpha(\phi(s)): I' \to R^2 \text{ is p.b.a.l.} \end{cases}$$

$$\therefore 1 = \left|\frac{d}{dt}\alpha(\phi(s))\right| \cdot |\phi'(s)| = |\alpha'(\phi(s))| \cdot |\phi'(s)|$$

we denote $\beta(s) = \alpha(\phi(s))$

2°,
$$\begin{cases} \beta'(s) = \alpha'(\phi(s)) \cdot \phi'(s) \\ \beta''(s) = \alpha''(\phi(s)) \cdot \phi'(s) + \alpha'(\phi(s)) \cdot \phi''(s) \end{cases}$$

$$\therefore k_\beta(s) = \langle \beta''(s), J(\beta'(s)) \rangle$$
$$= \langle (\phi')^2 \alpha'' + \phi'' \alpha', J(\alpha') \cdot \phi' \rangle$$
$$= (\phi')^3 \langle \alpha'', J(\alpha') \rangle \quad [\alpha' \perp J(\alpha')]$$
$$= \frac{\det(\alpha', \alpha'')}{|\alpha'|^3}$$

③ example: $\alpha(t): (-1, +\infty) \to R^2$, given by

$$\alpha(t) = \left(\frac{3t}{1+t^3}, \frac{3t^2}{1+t^3}\right)$$

Ans: 1°, $\alpha'(t) = \left(\frac{3-6t^3}{(1+t^3)^2}, \frac{6t-3t^4}{(1+t^3)^2}\right)$

$$\alpha''(t) = \left(\frac{18t^5-36t^2}{(1+t^3)^3}, \frac{6-42t^3+6t^6}{(1+t^3)^3}\right)$$

$2°$, $|\alpha'|^2 = \dfrac{9[1+4t^2-4t^3-4t^5+4t^6+t^8]}{(1+t^3)^4}$

$$\det(\alpha',\alpha'') = \frac{(3-6t^3)(6-42t^3+6t^6)-(6t-3t^4)(18t^5-36t^2)}{(1+t^3)^5}$$

$$= \frac{18(1+3t^3+3t^6+t^9)}{(1+t^3)^5}$$

$$= \frac{18(1+t^3)^3}{(1+t^3)^5} = \frac{18}{(1+t^3)^2}$$

$3°$, $k(t) = \dfrac{\det(\alpha',\alpha'')}{|\alpha'|^3} = \dfrac{2(1+t^3)^4}{3[1+4t^2-4t^3-4t^5+4t^6+t^8]^{\frac{3}{2}}}$