THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH4030 Differential Geometry
Solution of Assignment 1

1. (a)

$$\alpha(s) = \left(\frac{1}{3}(1+s)^{\frac{3}{2}}, \frac{1}{3}(1-s)^{\frac{3}{2}}, \frac{1}{\sqrt{2}}s\right)$$

$$\alpha'(s) = \left(\frac{1}{2}(1+s)^{\frac{1}{2}}, -\frac{1}{2}(1-s)^{\frac{1}{2}}, \frac{1}{\sqrt{2}}\right)$$

Since $|\alpha'(s)|^2 = \frac{1}{4}(1+s) + \frac{1}{4}(1-s) + \frac{1}{2} = 1$, we know that $\alpha(s)$ is p.b.a.l.

$$\alpha''(s) = \left(\frac{1}{4\sqrt{1+s}}, \frac{1}{4\sqrt{1-s}}, 0\right)$$

$$\begin{aligned}|\alpha''(s)|^2 &= \frac{1}{16(1+s)} + \frac{1}{16(1-s)} \\ &= \frac{1}{8(1-s^2)}\end{aligned}$$

So we have that

$$\begin{aligned}T &= \alpha'(s) \\ &= \left(\frac{1}{2}(1+s)^{\frac{1}{2}}, -\frac{1}{2}(1-s)^{\frac{1}{2}}, \frac{1}{\sqrt{2}}\right)\end{aligned}$$

$$\begin{aligned}N &= \frac{T'}{|T'|} \\ &= \frac{\alpha''}{|\alpha''|} \\ &= 2\sqrt{2}\sqrt{1-s^2}\left(\frac{1}{4\sqrt{1+s}}, \frac{1}{4\sqrt{1-s}}, 0\right) \\ &= \left(\frac{\sqrt{2(1-s)}}{2}, \frac{\sqrt{2(1+s)}}{2}, 0\right)\end{aligned}$$

$$\begin{aligned}B &= T \times N \\ &= \left(-\frac{\sqrt{1+s}}{2}, \frac{\sqrt{1-s}}{2}, \frac{\sqrt{2}}{2}\right)\end{aligned}$$

Then we compute κ and τ,

$$\kappa = |T^{'}| = \frac{1}{2\sqrt{2(1-s^2)}}$$

$$\begin{aligned}\tau &= < B^{'}, N > \\ &= \left\langle \left(-\frac{1}{4\sqrt{1+s}}, -\frac{1}{4\sqrt{1-s}}, 0\right), \left(\frac{\sqrt{2(1-s)}}{2}, \frac{\sqrt{2(1+s)}}{2}, 0\right)\right\rangle \\ &= -\frac{\sqrt{2}}{8}\left(\sqrt{\frac{1-s}{1+s}} + \sqrt{\frac{1+s}{1-s}}\right) \\ &= -\frac{\sqrt{2}}{8}\left(\frac{1-s+1+s}{\sqrt{1-s^2}}\right) \\ &= -\frac{\sqrt{2}}{4\sqrt{1-s^2}}\end{aligned}$$

(b)

$$\alpha(t) = \left(\sqrt{1+t^2}, t, \log t + \sqrt{1+t^2}\right)$$

$$\begin{aligned}\alpha^{'}(t) &= \left(\frac{t}{\sqrt{1+t^2}}, 1, \frac{t+\sqrt{1+t^2}}{t(\sqrt{1+t^2}) + (1+t^2)}\right) \\ &= \left(\frac{t}{\sqrt{1+t^2}}, 1, \frac{1}{\sqrt{1+t^2}}\right)\end{aligned}$$

Since $|\alpha^{'}(t)|^2 = 2$, we set $\beta(s) = \alpha(\frac{s}{\sqrt{2}})$.
Then we have

$$\begin{aligned}\beta^{'}(s) &= \left(\frac{\frac{s}{2}}{\sqrt{1+\frac{s^2}{2}}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}\sqrt{1+\frac{s^2}{2}}}\right) \\ &= \left(\frac{s}{\sqrt{2}\sqrt{2+s^2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2+s^2}}\right)\end{aligned}$$

$$\beta^{''}(s) = \left(\frac{\sqrt{2}}{(2+s^2)^{\frac{3}{2}}}, 0, -\frac{s}{(2+s^2)^{\frac{3}{2}}}\right)$$

$$|\beta^{''}(s)|^2 = \frac{1}{(2+s^2)^2}$$

So we have that

$$T = \beta^{'}(s)$$
$$= \left(\frac{s}{\sqrt{2}\sqrt{2+s^2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2+s^2}}\right)$$

$$N = \frac{T^{'}}{|T^{'}|}$$
$$= \frac{\beta^{''}}{|\beta^{''}|}$$
$$= \left(\frac{\sqrt{2}}{\sqrt{2+s^2}}, 0, -\frac{s}{\sqrt{2+s^2}}\right)$$

$$B = T \times N$$
$$= \left(-\frac{s}{\sqrt{2}\sqrt{2+s^2}}, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2+s^2}}\right)$$

Then we compute κ and τ,

$$\kappa = |T^{'}|$$
$$= \frac{1}{2+s^2}$$

$$\tau =< B^{'}, N >$$
$$= \left\langle \left(-\frac{\sqrt{2}}{(2+s^2)^{\frac{3}{2}}}, 0, \frac{s}{(2+s^2)^{\frac{3}{2}}}\right), \left(\frac{\sqrt{2}}{\sqrt{2+s^2}}, 0, -\frac{s}{\sqrt{2+s^2}}\right)\right\rangle$$
$$= \frac{-2-s^2}{(2+s^2)^2}$$
$$= -\frac{1}{2+s^2}$$

2. (a)

$$\alpha^{'}(t) = (1, f^{'}(t))$$
$$|\alpha^{'}(t)|^2 = 1 + (f^{'}(t))^2 > 1$$

So $\alpha(t)$ is regular.

(b)

$$\begin{aligned} length &= \int_a^b |\alpha^{'}(t)| dt \\ &= \int_a^b \sqrt{1 + (f'(t))^2} dt \end{aligned}$$

(c)

$$\alpha^{''}(t) = (0, f^{''}(t))$$

$$\begin{aligned} \kappa &= \frac{det(\alpha^{'}(t), \alpha^{''}(t))}{|\alpha'(t)|^3} \\ &= \frac{f^{''}(t)}{(1 + (f'(t))^2)^{\frac{3}{2}}} \end{aligned}$$

3. (a)

$$\alpha'(t) = (1, \sinh t)$$
$$|\alpha'(t)|^2 = 1 + \sinh^2 t = \cosh^2 t$$

$$\begin{aligned} length &= \int_0^b \cosh t dt \\ &= \sinh(b) - \sinh(0) \\ &= \sinh(b) \end{aligned}$$

(b)

$$\begin{aligned} s(t) &= \int_0^t |\alpha'(x)| dx \\ &= \int_0^t \cosh x dx \\ &= \sinh(t) \end{aligned}$$

So we have $t = \sinh^{-1} s$.

$$\begin{aligned} \beta(s) &= \alpha(\sinh^{-1} s) \\ &= \big(\sinh^{-1} s, \cosh(\sinh^{-1} s)\big) \\ &= \left(\sinh^{-1} s, \sqrt{[\sinh(\sinh^{-1} s)]^2 + 1}\right) \\ &= \left(\sinh^{-1} s, \sqrt{s^2 + 1}\right) \end{aligned}$$

And $s_0 = length = \sinh(b)$

(c)

$$\begin{aligned} \kappa_\beta &= < \beta''(s), J(\beta'(s)) > \\ &= \left\langle \left(-\frac{s}{(s^2+1)^{\frac{3}{2}}}, \frac{1}{(s^2+1)^{\frac{3}{2}}}\right), \left(-\frac{s}{\sqrt{s^2+1}}, \frac{1}{\sqrt{s^2+1}}\right) \right\rangle \\ &= \frac{s^2+1}{(s^2+1)^2} \\ &= \frac{1}{s^2+1} \end{aligned}$$

4. (a)

$$\alpha^{'}(t) = (-\sin t + \frac{1}{\sin t}, \cos t)$$

$$|\alpha^{'}(t)|^2 = \frac{\cos^2 t}{\sin^2 t}$$

So the arc length of $\alpha(t), t \in (\frac{\pi}{2}, \frac{\pi}{2} + s)$ is

$$\begin{aligned} length &= \int_{\frac{\pi}{2}}^{\frac{\pi}{2}+s} -\frac{\cos t}{\sin t} dt \\ &= -\log(\sin(\frac{\pi}{2} + s)) + \log(\sin(\frac{\pi}{2})) \\ &= -\log(\cos s) \end{aligned}$$

(b)

$$\alpha^{''}(t) = \left(-\cos t - \frac{\cos t}{\sin^2 t}, -\sin t\right)$$

The signed curvature of the tractrix is

$$\begin{aligned} \kappa &= \frac{det(\alpha^{'}(t), \alpha^{''}(t))}{|\alpha^{'}(t)|^3} \\ &= \frac{\sin^2 t - 1 + \cos^2 t + \frac{\cos^2 t}{\sin^2 t}}{-\frac{\cos^3 t}{\sin^3 t}} \\ &= -\tan t \end{aligned}$$

(c) At the point $\alpha(t)$, the equation of the tangent line is

$$(x(s), y(s)) = \alpha(t) + s\alpha^{'}(t)$$

When $y(s) = 0$,

$$\sin t + s \cos t = 0$$

$$s = -\tan t$$

So the length between $\alpha(t)$ and $(x(-\tan t), 0)$ is

$$|s\alpha^{'}(t)| = |-\tan t \frac{\cos t}{\sin t}| = 1$$

5. Consider the function $f(s) = |\alpha(s)|^2 = <\alpha(s), \alpha(s)>$.
Since $f(s)$ attains a maximum at $s = s_0$, we have

$$\begin{aligned} 0 &\geq f^{''}(s_0) \\ &= 2 < \alpha(s_0), \alpha^{''}(s_0) > +2 < \alpha^{'}(s_0), \alpha^{'}(s_0) > \\ &= 2 < \alpha(s_0), \alpha^{''}(s_0) > +2 \end{aligned}$$

So we get $< \alpha(s_0), \alpha^{''}(s_0) > \leq -1$.

$$1 \leq | < \alpha(s_0), \alpha^{''}(s_0) > | \leq |\alpha(s_0)||\alpha^{''}(s_0)| = R|\alpha^{''}(s_0)|$$

By the assumption that $\alpha(t)$ is p.b.a.l,

$$\kappa(s_0) = |\alpha^{''}(s_0)| \geq \frac{1}{R}$$

6. When $\alpha(a) = \alpha(b)$, the result is trivial.
   So assume that $\alpha(a) \neq \alpha(b)$, and let $\vec{n} = \dfrac{\alpha(b) - \alpha(a)}{|\alpha(b) - \alpha(a)|}$.
   By the fundamental theorem of calculus, we have

$$\alpha(b) - \alpha(a) = \int_a^b \alpha^{'}(t)dt$$

   Then

$$\begin{aligned} |\alpha(b) - \alpha(a)| &=< \alpha(b) - \alpha(a), \vec{n} > \\ &=< \int_a^b \alpha^{'}(t)dt, \vec{n} > \\ &= \int_a^b < \alpha^{'}(t), \vec{n} > dt \\ &\leq \int_a^b |\alpha^{'}(t)||\vec{n}|dt \\ &= \int_a^b |\alpha^{'}(t)|dt \\ &= length(\alpha) \end{aligned}$$

7. Let $\alpha(t) : I \mapsto R^3$ be a regular curve.
Let $\beta(s) = \alpha(\phi(s))$ be p.b.a.l. where $t = \phi(s)$ is an increasing function.
By the definition, $\tau =< B^{'}, N >= - < B, N^{'} >$.

We first compute the relation between $\phi(s)$ and $\alpha(t)$.
Since $\beta(s)$ is p.b.a.l, we have

$$1 = |\beta^{'}(s)| = |\alpha^{'}(t)|\phi^{'}(s)$$

$$\phi^{'}(s) = \frac{1}{|\alpha^{'}(t)|}$$

Differentiate again, we get

$$\begin{aligned}
\phi^{''}(s) &= (< \alpha^{'}(t), \alpha^{'}(t) >^{-\frac{1}{2}})^{'} \\
&= -\frac{1}{2} < \alpha^{'}(t), \alpha^{'}(t) >^{-\frac{3}{2}} (2 < \alpha^{'}(t), \alpha^{''}(t)\phi^{'}(s) >) \\
&= -\frac{\phi^{'}(s)}{|\alpha^{'}(t)|^3} < \alpha^{'}(t), \alpha^{''}(t) > \\
&= -\frac{1}{|\alpha^{'}(t)|^4} < \alpha^{'}(t), \alpha^{''}(t) >
\end{aligned}$$

Then we compute the frame $\{T, N, B\}$ and κ,

$$\begin{aligned}
T &= \beta^{'}(s) = \alpha^{'}(t)\phi^{'}(s) \\
T^{'} &= \alpha^{''}(t)(\phi^{'}(s))^2 + \alpha^{'}(t)\phi^{''}(s) \\
\kappa^2 &= |T^{'}|^2 \\
&= |\alpha^{''}(t)|^2(\phi^{'}(s))^4 + 2(\phi^{'}(s))^2\phi^{''}(s) < \alpha^{'}(t), \alpha^{''}(t) > + |\alpha^{'}(t)|^2(\phi^{''}(s))^2 \\
&= \frac{|\alpha^{''}(t)|^2}{|\alpha^{'}(t)|^4} - 2\frac{1}{|\alpha^{'}(t)|^2}\frac{< \alpha^{'}(t), \alpha^{''}(t) >^2}{|\alpha^{'}(t)|^4} + \frac{< \alpha^{'}(t), \alpha^{''}(t) >^2}{|\alpha^{'}(t)|^8}|\alpha^{'}(t)|^2 \\
&= \frac{|\alpha^{''}(t)|^2}{|\alpha^{'}(t)|^4} - \frac{< \alpha^{'}(t), \alpha^{''}(t) >^2}{|\alpha^{'}(t)|^6} \\
&= \frac{|\alpha^{''}(t)|^2}{|\alpha^{'}(t)|^4} - \frac{|\alpha^{'}(t)|^2|\alpha^{''}(t)|^2 \cos^2\theta}{|\alpha^{'}(t)|^6} \text{where } \theta \text{ is the angle between } \alpha^{'}(t) \text{ and } \alpha^{''}(t) \\
&= \frac{|\alpha^{''}(t)|^2}{|\alpha^{'}(t)|^4}\sin^2\theta \\
&= \frac{|\alpha^{'}(t) \times \alpha^{''}(t)|^2}{|\alpha^{'}(t)|^6} \\
N &= \frac{T^{'}}{\kappa} \\
&= \frac{(\phi^{'}(s))^2}{\kappa}\alpha^{''}(t) + \frac{\phi^{''}(s)}{\kappa}\alpha^{'}(t) \\
B &= T \times N \\
&= \frac{(\phi^{'}(s))^3}{\kappa}\alpha^{'}(t) \times \alpha^{''}(t)
\end{aligned}$$

Finally,

$$
\begin{aligned}
N^{'} &= \left(\frac{T^{'}}{|T^{'}|}\right)^{'} \\
&= \frac{T^{''}|T^{'}| - T^{'}\frac{<T^{'},T^{''}>}{|T^{'}|}}{|T^{'}|^2} \\
&= \frac{T^{''}|T^{'}|^2 - T^{'} < T^{'}, T^{''} >}{|T^{'}|^3}
\end{aligned}
$$

Since $B = \dfrac{(\phi^{'}(s))^3}{\kappa}\alpha^{'}(t) \times \alpha^{''}(t)$, to compute $\tau = - < B, N^{'} >$, we only need to compute the term $\alpha^{'''}$ in $N^{'}$.

$$
\begin{aligned}
\tau &= - < B, N^{'} > \\
&= -\frac{(\phi^{'}(s))^3}{\kappa}\frac{1}{|T^{'}|}(\phi^{'}(s))^3 < \alpha^{'}(t) \times \alpha^{''}(t), \alpha^{'''}(t) > \\
&= -\frac{(\phi^{'}(s))^6}{\kappa^2} < \alpha^{'}(t) \times \alpha^{''}(t), \alpha^{'''}(t) > \\
&= -\frac{< \alpha^{'}(t) \times \alpha^{''}(t), \alpha^{'''}(t) >}{|\alpha^{'}(t) \times \alpha^{''}(t)|^2}
\end{aligned}
$$

8. (a) Let $\alpha(t_1) = \alpha(t_2)$,

$$\frac{3t_1}{1+t_1^3} = \frac{3t_2}{1+t_2^3}$$
$$\frac{3t_1^2}{1+t_1^3} = \frac{3t_2^2}{1+t_2^3}$$

If one of t_1, t_2 is zero, then $t_1 = t_2 = 0$.
If none of t_1, t_2 is zero,

$$\frac{1}{t_1} = \frac{1}{t_2}$$
$$t_1 = t_2$$

So $\alpha(t)$ is one to one.

(b) $\alpha(0) = (0, 0)$
As $t \to \infty, \alpha(t) = \left(\frac{3t}{1+t^3}, \frac{3t^2}{1+t^3}\right) \to (0, 0)$.
So $\alpha(t) : (-1, \infty) \mapsto R^2$ is not a homeomorphism onto its image.

9. (a) $'' \Rightarrow ''$ W.L.O.G, we may assume that $\alpha(s) \subset B_R(0)$.

$$< \alpha, \alpha >= R$$
$$< \alpha, \alpha^{'} >= 0$$

we get $< \alpha, T >= 0$.

$$< \alpha, \alpha^{'} >^{'} = 0$$
$$< \alpha, \alpha^{''} > + < \alpha^{'}, \alpha^{'} >= 0$$
$$< \alpha, \alpha^{''} > +1 = 0$$

we get $< \alpha, \kappa N >= -1$.

$$(< \alpha, \alpha^{''} > +1)^{'} = 0$$
$$< \alpha^{'}, \alpha^{''} > + < \alpha, \alpha^{'''} >= 0$$
$$< \alpha, \alpha^{'''} >= - < \alpha^{'}, \alpha^{''} >= -\frac{1}{2} < \alpha^{'}, \alpha^{'} >^{'} = 0$$

we get $< \alpha, \alpha''' >= 0$.

$$\begin{aligned} \alpha''' &= (\alpha'')^{'} \\ &= (\kappa N)^{'} \\ &= \kappa^{'} N + \kappa N^{'} \\ &= \kappa^{'} N + \kappa(-\kappa T - \tau B) \\ &= -\kappa^2 T + \kappa^{'} N - \kappa \tau B \end{aligned}$$

So we have

$$\begin{aligned} < \alpha, T > &= 0 \\ < \alpha, N > &= -\frac{1}{\kappa} \\ < \alpha, -\kappa^2 T + \kappa^{'} N - \kappa \tau B > &= 0 \end{aligned}$$

This gives us $\alpha = -\frac{1}{\kappa} N - \frac{\kappa^{'}}{\kappa^2 \tau} B = -\frac{1}{\kappa} N + \frac{1}{\tau}(\frac{1}{\kappa})' B$.

$$\alpha^{'} = T$$
$$-(\frac{1}{\kappa})' N - \frac{1}{\kappa} N' + (\frac{1}{\tau}(\frac{1}{\kappa})')' B + \frac{1}{\tau}(\frac{1}{\kappa})' B' = T$$
$$\frac{\kappa^{'}}{\kappa^2} N - \frac{1}{\kappa}(-\kappa T - \tau B) + (\frac{1}{\tau}(\frac{1}{\kappa})')' B + \frac{1}{\tau}(\frac{1}{\kappa})'(\tau N) = T$$
$$\left(\frac{\tau}{\kappa} + (\frac{1}{\tau}(\frac{1}{\kappa})')'\right) B = 0$$
$$\frac{\tau}{\kappa} + (\frac{1}{\tau}(\frac{1}{\kappa})')' = 0$$

(b) $'' \Leftarrow ''$ Given $\frac{\tau}{\kappa} + (\frac{1}{\tau}(\frac{1}{\kappa})')' = 0$.
Define $P(s) = -\frac{1}{\kappa}N + \frac{1}{\tau}(\frac{1}{\kappa})'B$.
Then by the previous computation and the assumption,

$$P'(s) = T = \alpha'(s)$$

$$(P(s) - \alpha(s))' = 0$$

$\alpha(s) - P(s) = P_0$ for some fixed point P_0.

$$\begin{aligned} < \alpha(s) - P_0, \alpha(s) - P_0 >' &= 2 < \alpha(s) - P_0, \alpha'(s) > \\ &= 2 < P(s), T > \\ &= 2 < -\frac{1}{\kappa}N + \frac{1}{\tau}(\frac{1}{\kappa})'B, T > \\ &= 0 \end{aligned}$$

So $|\alpha(s) - P_0| = R$ for some positive constant R. Which means $\alpha(s)$ lies on some sphere.