

THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2050C Mathematical Analysis I

Tutorial 7 (March 18)

Definition (Contractive Sequences). We say that a sequence (x_n) of real numbers is **contractive** if there exists a constant C , $0 < C < 1$, such that

$$|x_{n+2} - x_{n+1}| \leq C|x_{n+1} - x_n| \quad \text{for all } n \in \mathbb{N}. \quad (\#)$$

The number C is called the **constant** of the contractive sequence.

Remark. Do not confuse $(\#)$ with the following condition:

$$|x_{n+2} - x_{n+1}| < |x_{n+1} - x_n| \quad \text{for all } n \in \mathbb{N}. \quad (\#\#)$$

For example, (\sqrt{n}) satisfies $(\#\#)$ but it is not contractive.

Theorem. *Every contractive sequence is a Cauchy sequence, and therefore is convergent.*

Example 1. (Sequence of Fibonacci Fractions) Consider the Fibonacci fractions $x_n := f_n/f_{n+1}$, where (f_n) is the Fibonacci sequence defined by $f_1 = f_2 = 1$ and $f_{n+2} := f_{n+1} + f_n$, $n \in \mathbb{N}$. Show that the sequence (x_n) converges to $1/\varphi$, where $\varphi := (1 + \sqrt{5})/2$ is the Golden Ratio.

Example 2. Show that if (x_n) is unbounded, then there exists a subsequence (x_{n_k}) such that $\lim(1/x_{n_k}) = 0$.

Solution. As (x_n) is unbounded, we have $\forall M > 0, \exists n \in \mathbb{N}$ such that $|x_n| > M$.

Pick $n_1 \in \mathbb{N}$ such that $|x_{n_1}| > 1$.

Then pick $n_2 \in \mathbb{N}$ such that $|x_{n_2}| > \max\{2, |x_1|, |x_2|, \dots, |x_{n_1}|\}$. So $|1/x_{n_2}| < 1/2$ and $n_2 > n_1$.

Suppose $n_1 < n_2 < \dots < n_k$ are chosen. Pick $n_{k+1} \in \mathbb{N}$ such that $|x_{n_{k+1}}| > \max\{k + 1, |x_1|, |x_2|, \dots, |x_{n_k}|\}$. So $|1/x_{n_{k+1}}| < 1/(k + 1)$ and $n_{k+1} > n_k$.

Continue in this way, we obtain a subsequence (x_{n_k}) of (x_n) such that

$$|1/x_{n_k}| < 1/k \quad \text{for all } k \in \mathbb{N}.$$

Now $\lim(1/x_{n_k}) = 0$ follows immediately from Squeeze Theorem. ◀

Classwork

- Let $x_n := \sqrt{n}$. Show that (x_n) satisfies $\lim|x_{n+1} - x_n| = 0$, but that it is not a Cauchy sequence by definition.
- Let (x_n) be a sequence of real numbers defined by

$$\begin{cases} x_1 = 1, & x_2 = 2, \\ x_{n+2} := \frac{1}{3}(2x_{n+1} + x_n) & \text{for all } n \in \mathbb{N}. \end{cases}$$

Show that (x_n) is convergent and find its limit.