THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH2050C Mathematical Analysis I

Tutorial 6 (March 11)

Monotone Convergence Theorem. *A monotone sequence of real number is convergent if and only if it is bounded. Furthermore,*

(a) If (x_n) is a bounded increasing sequence, then $\lim(x_n) = \sup\{x_n : n \in \mathbb{N}\}$.

(b) If (y_n) is a bounded decreasing sequence, then $\lim(y_n) = \inf\{y_n : n \in \mathbb{N}\}$.

Example 1. Let $Z = (z_n)$ be the sequence of real numbers defined by

$$z_1 \coloneqq 1, \quad z_{n+1} \coloneqq \sqrt{2z_n} \quad \text{for } n \in \mathbb{N}.$$

Show that $\lim(z_n) = 2$.

Example 2 (*Euler number e*)**.** Let $e_n \coloneqq (1 + 1/n)^n$ for $n \in \mathbb{N}$. Show that the sequence $E = (e_n)$ is bounded and increasing, hence convergent. The limit of this sequence is called the *Euler number*, and it is denoted by e.

Definition. Let $X = (x_n)$ be a sequence of real numbers and let $n_1 < n_2 < \cdots < n_k < \cdots$ be a **strictly increasing** sequence of natural numbers. Then the sequence $X' = (x_{n_k})$ given by

$$(x_{n_1}, x_{n_2}, \ldots, x_{n_k}, \ldots)$$

is called a **subsequence** of X.

Theorem. *If a sequence $X = (x_n)$ of real numbers converges to a real number x, then any subsequence $X' = (x_{n_k})$ of X also converges to x.*

Example 3. By considering subsequences, deduce the following limits.

(a) $\lim(b^n) = 0$ if $0 < b < 1$.

(b) $\lim(c^{1/n}) = 1$ if $c > 1$.

Classwork

1. Establish the convergence or the divergence of the sequence (x_n), where

$$x_n \coloneqq \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \qquad \text{for } n \in \mathbb{N}.$$

2. Let $y_1 \coloneqq \sqrt{p}$, where $p > 0$, and $y_{n+1} \coloneqq \sqrt{p + y_n}$ for $n \in \mathbb{N}$. Show that (y_n) converges and find the limit. (Hint: $1 + 2\sqrt{p}$ is one upper bound.)