THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH2050C Mathematical Analysis I

Tutorial 4 (February 26)

Definition. A sequence $X = (x_n)$ in $\mathbb{R}$ is said to **converge** to $x \in \mathbb{R}$, or x is said to be a **limit** of (x_n), if for every $\varepsilon > 0$ there exists a natural number $K(\varepsilon)$ such that for all $n \geq K(\varepsilon)$, the terms x_n satisfy $|x_n - x| < \varepsilon$.

Remark. (1) The notion of limit depends only on the tail of the sequence.

(2) " $|x_n - x| < \varepsilon$" could be replaced by " $|x_n - x| \leq \varepsilon$".

(3) The definition does not tell you how to find the limit.

(4) Notations: $\lim X = x, \quad \lim(x_n) = x, \quad \lim_n x_n = x, \quad \lim_{n\to\infty} x_n = x.$

Example 1. Use the definition of the limit of a sequence to show $\lim \left(\dfrac{n^2 - n}{2n^2 + 3}\right) = \dfrac{1}{2}$.

Solution.

1. Fix an arbitrary $\varepsilon > 0$. It cannot be changed once fixed.

Let $\varepsilon > 0$ be given.

2. Find a useful estimate for $|x_n - x|$.

For $n \geq 1$,

$$\begin{aligned}\left|\frac{n^2 - n}{2n^2 + 3} - \frac{1}{2}\right| &= \left|\frac{2n^2 - 2n - 2n^2 - 3}{2(2n^2 + 3)}\right| = \frac{2n + 3}{2(2n^2 + 3)} \\ &\leq \frac{2n + 3}{n^2} \\ &\leq \frac{2n + 3n}{n^2} = \frac{5}{n}.\end{aligned}$$

Do not try to solve $\dfrac{2n + 3}{2(2n^2 + 3)} < \varepsilon$ directly.

3. Find $K = K(\varepsilon) \in \mathbb{N}$ such that the above estimate is less than ε whenever $n \geq K$.

By Archimedean Property, there is $K \in \mathbb{N}$ such that $K > 1/\varepsilon$.

4. Complete the argument.

Now, for all $n \geq K$, we have

$$\left|\frac{n^2 - n}{2n^2 + 3} - \frac{1}{2}\right| \leq \frac{5}{n} \leq \frac{5}{K} < \varepsilon.$$

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Example 2. Use the definition to show the following limits.

(a) $\lim \dfrac{1}{n^2+1} = 0.$

(b) $\lim \dfrac{3n+2}{n+1} = 3.$

(c) $\lim(\sqrt{n+1} - \sqrt{n}) = 0.$

Example 3. Let (x_n) be a sequence given by $1 + (-1)^n$. Show that (x_n) is divergent.

Solution. Suppose on the contrary that (x_n) converges. Assume $\lim(x_n) = \ell \in \mathbb{R}$. Then for $\varepsilon_0 = 1/2$, there exists $K \in \mathbb{N}$ such that $|x_n - \ell| < \varepsilon_0$ whenever $n \geq K$. In particular,

$$|x_K - x_{K+1}| = |(x_K - \ell) - (x_{K+1} - \ell)| \leq |x_K - \ell| + |x_{K+1} - \ell| < \varepsilon_0 + \varepsilon_0 = 1. \qquad (*)$$

However,

$$|x_K - x_{K+1}| = |(1 + (-1)^K) - (1 + (-1)^{K+1})| = |(-1)^K - (-1)^{K+1}| = 2,$$

contradicting $(*)$. Thus (x_n) is divergent. ◀

Classwork

1. Use the definition of the limit of a sequence to establish the following limits.

   (a) $\lim \left(\dfrac{1}{n} - \dfrac{1}{n+1}\right) = 0$

   (b) $\lim \dfrac{2n^2+1}{(n+1)^2} = 2.$