# MATH 2050C Mathematical Analysis I
# 2019-20 Term 2

## Hard problems in Chapter 5

### 5.1-4(c)

The range of $\sin x$ is $[-1,1]$. So the values of $[\![\sin x]\!]$ can be taken are $-1, 0, 1$. By solving $\sin x < 0, \sin x < 1, \sin x = -1, \sin x = 1$, we get

$$[\![\sin x]\!] = \begin{cases} 1, & x = 2\pi k + \frac{\pi}{2}, k \in \mathbb{Z} \\ 0, & x \in [2\pi k, \frac{\pi}{2} + 2\pi k) \cup (\frac{\pi}{2} + 2\pi k, \pi + 2\pi k], k \in \mathbb{Z} \\ -1, & x \in (2\pi k - \pi, 2\pi k), k \in \mathbb{Z} \end{cases}$$

So in the interior of those intervals, $h(x)$ is continuous since $h(x)$ is constant. But at the boundary point, such as $2\pi k, \frac{\pi}{2}+2\pi k, 2\pi k+\pi$, they are not continuous since it will take different intergers at any intervals containing those points. So the continous points of $h(x)$ are $(2\pi k - \pi, 2\pi k) \cup (2\pi k, \frac{\pi}{2} + 2\pi k) \cup (\frac{\pi}{2} + 2\pi k, \pi + 2\pi k)$, $k \in \mathbb{Z}$.

### 5.1-9

**(a)**. If $f$ is continous at $c \in A$, then for any $\epsilon > 0$, we can find $\delta > 0$, such that for any $x \in (c-\delta, c+\delta) \cap A$, we have

$$|f(x) - f(c)| < \epsilon$$

since $A \subset B$, we know for any $x \in (c-\delta, c+\delta) \cap B$, we still have

$$|f(x) - f(c)| < \epsilon$$

And since $f, g$ agrees on $B$, so we get

$$|g(x) - g(c)| < \epsilon$$

Hence, $g$ is continuous at $c$.
**(b)**. Choose $f(x)$ as

$$f(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$$

And we choose $A = \mathbb{R}$, $B = [0, +\infty)$. So on $B$, $g$ is a constant function 1 and it is continuous at 0. But clearly, $f(x)$ is not continuous at $x = 0$.

## 5.1-14

We proof that, given any open interval $(a, b)$, the function $k(x)$ is unbounded on this interval. This means for any $M > 0$, we need to find some $x_0 \in (a, b)$ such that $|k(x_0)| > M$. We note that there are only finite number of rationals with denominator less than $M$. (Note there are at most $\frac{b-a+1}{n}$ rationals with denominator equal to $n$.) So we can find a rational number $x_0 = \frac{p_0}{q_0}$ with $q_0 > M$ and $p_0, q_0$ have no common factor except 1. Hence we have $k(x_0) = q_0 > M$. So we get $k(x)$ is not bounded on this interval.

## 5.2-9

Let's take arbitrary $x_0 \in \mathbb{R}$ first. We will show that for any $\epsilon > 0$, we have $|h(x_0)| < \epsilon$.

Indeed, for any $\epsilon > 0$, we can find $\delta > 0$ such that $|h(x) - h(x_0)| < \epsilon$ for any $|x - x_0| < \delta$ since $h(x)$ is continous at $x_0$. We claim that we can find $m, n$ such that $\frac{m}{2^n} \in (x_0 - \delta, x_0 + \delta)$. If so, we have $h(\frac{m}{2^n}) = 0$, and hence $|h(x_0)| = |h(x_0) - h(\frac{m}{2^n})| < \epsilon$ and we finish the proof.

Proof of claim. We first choose $2^n \geq 1 + n > \frac{1}{\delta}$, that is $\frac{1}{2^n} < \delta$ and consider the set $A = \{m \in \mathbb{Z} : \frac{m}{2^n} \leq x_0 - \delta\}$. This set has an upper bound (i.e. $2^n(x_0 - \delta)$), so we can take $u = \sup A$. It can be easily showed that $u$ is still an integer. (If not, the integer part of $u$ is also an upper bound leading to a contradiction). Now we choose $m = u + 1$. Clearly $\frac{m}{2^n} > x_0 - \delta$ but we also note $u \in A$ (If not, then $u - 1$ will be an upper bound), we have $\frac{m-1}{2^n} \leq x_0 - \delta \implies \frac{m}{2^n} \leq x_0 - \delta + \frac{1}{2^n} < x_0 + \delta$. This means $\frac{m}{2^n} \in (x_0 - \delta, x_0 + \delta)$ and finish the proof of claim.

## 5.2-12

Note that $f(0) = f(0) + f(0)$ which will imply $f(0) = 0$. And note that $f(x) = f(x - x_0) + f(x_0)$. So if we take $x \to x_0$, then we have $x - x_0 \to 0$. The right side $f(x) \to f(x_0)$ since $f(x)$ is continuous at $x_0$. Then for the right side, we will have $\lim_{x\to 0} f(x) + f(x_0)$. Combining this we get

$$\lim_{x\to 0} f(x) = 0$$

which indicates $f(x)$ is continous at $x = 0$. Again, for any $y_0 \in \mathbb{R}$, we have

$$f(x) = f(x - y_0) + f(y_0)$$

Then taking $x \to y_0$, we will get

$$\lim_{x\to y_0} f(x) = \lim_{x\to y_0} f(x - y_0) + f(y_0) = \lim_{x\to 0} f(x) + f(y_0) = 0 + f(y_0) = f(y_0)$$

This shows $f$ is continous at $y_0$. Hence $f$ is continous at $y_0$. Since $y_0$ is arbitrary, we get $f$ is indeed continuous at every point of $\mathbb{R}$.

## 5.3-4

Given any polynomial $p(x)$ of odd degree, without loss of generality, denote

$$p(x) := \sum\nolimits_{i=0}^{2n+1} a_i x^i, \quad a_{2n+1} = 1, n \in \mathbb{N}.$$

Also denote $M := \max\{|a_0|, \cdots |a_{2n}|\}$. For $x > \max\{2nM + 1, 1\}$, we have

$$\begin{aligned} p(x) &= \sum\nolimits_{i=0}^{2n+1} a_i x^i \\ &\geq x^{2n+1} - M(x^{2n} + \cdots + x + 1) \\ &\geq x^{2n+1} - 2nMx^{2n} \quad (x^k \geq x^l, \forall x \geq 1, k \leq l \in \mathbb{N}) \\ &= x^{2n}(x - 2nM) \\ &> (2nM + 1)^{2n} > 0. \end{aligned}$$

and similarly $p(x) < 0$ for $x < -\max\{2nM + 1, 1\} = \min\{-2nM - 1, -1\}$. Set $R_0 = \max\{2nM + 1, 1\}$ so that $p(R_0)p(-R_0) < 0$. We deduce that there exists at least one real root for $p(x)$ on $[-R_0, R_0]$ by Theorem 5.3.5.

## 5.3-15

For open intervals, there are three kind of intervals as following

- $(a, b)$ with $a < b \leq 0$.
- $(a, b)$ with $a < 0 < b$.
- $(a, b)$ with $0 \leq a < b$.

For the first kind $I = (a, b)$ with $a < b \leq 0$, we have $f(I) = (b^2, a^2)$, which is an open intervals.

For the second kind $I = (a, b)$, $a < 0 < b$, we have $f(I) = [0, \max\{a, b\}]$, which is closed.

For the third kind, $I = (a, b)$, $0 \leq a < b$, we have $F(I) = (a^2, b^2)$, which is an open intervals. So $f(I)$ is an open intervals if and only if open intervals $I$ which does not contains 0.

For the closed intervals $I$, $f(I)$ will always be a closed intervals by Theorem 5.3.9.

## 5.3-17

We proof the following claim first.

If $f : [0, 1] \to \mathbb{R}$ is continous and has two different values $a, b$ with $a < b$, then $f$ cannot has only rational (or irrational) values.

Indeed, suppose $f(x_1) = a, f(x_2) = b$, then for any $c \in (a, b)$, by Intermediate Value Theorem, we can find $x_0 \in [x_1, x_2] or [x_2, x_1]$ such that $f(x_0) = c$. By Density Theorem, we can always find a rational number in $(a, b)$, and we can also find a irrational number in $(a, b)$. So this means $f$ has to take rational values and irrational values. So the claim is proofed.

So from this claim, we can see that if $f$ is not a constant, $f$ will take at least two different values and we can write these two vaules as $a, b$ with $a < b$ and hence $f$ has to take both rational and irrational values.

Hence if $f$ has only rational (irrational) values, $f$ has to be a constant.

## 5.4-12

Since $f$ is uniformly continuous on $[a, \infty)$, given $\varepsilon > 0$, there exists $\delta_1 > 0$ so that if $x, u \in [a, \infty)$ and $|x - u| < \delta_1$, then

$$|f(x) - f(u)| < \varepsilon.$$

Since $f$ is continuous on $[0, a+1]$, $f$ is uniformly continuous on $[0, a+1]$ by Theorem 5.4.3. Given $\varepsilon > 0$, there exists $\delta_2 > 0$ so that if $x, u \in [0, a+1]$ and $|x - u| < \delta_2$, then

$$|f(x) - f(u)| < \varepsilon.$$

Denote $\delta = \min\{\delta_1, \delta_2, 1\}$. Note that either $x, u \in [0, a+1]$ or $x, u \in [a, \infty)$ for any $x, u \in [0, \infty)$ with $|x - u| < \delta$. Thus $|f(x) - f(u)| < \varepsilon$ in either case.

## 5.4-14

For any $x \in \mathbb{R}$, $x \in [k_x p, k_x p + p)$ for some $k_x \in \mathbb{Z}$ and $x - k_x p \in [0, p)$, since $\mathbb{R} = \cup_{k \in \mathbb{Z}} [kp, kp + p)$. Denote $M = \sup\{|f(x)|, x \in [0, p]\}$. $M < \infty$ since $f$ is continuous and bounded on $[0, p]$. We have

$$|f(x)| = |f(x - k_x p)| \leq M, \quad \forall x \in \mathbb{R},$$

where the periodicity of $f$ is applied. We deduce that $f$ is bounded on $\mathbb{R}$.
To show the uniform continuity, first notice that $f$ is uniformly continuous on $[0, 2p]$. Given $\varepsilon > 0$, there exists $\delta_0(\varepsilon) > 0$ so that if $x, u \in [0, 2p]$ satisfying $|x - u| < \delta_0$, then $|f(x) - f(u)| < \varepsilon$. Now we show the uniform continuity on $\mathbb{R}$. Given $\varepsilon > 0$, denote $\delta = \min\{p, \delta_0(\varepsilon)\}$. Without loss of generality, we assume $x \leq u$. For any $x, u \in \mathbb{R}$ satisfying $|x - u| < \delta$, there are two cases.

(i) $u \in [k_x p, k_x p + p)$. Then $x - k_x p, u - k_x p \in [0, p]$ and $|(x - k_x p) - (u - k_x p)| = |x - u| < \delta$. Thus $|f(x) - f(u)| = |f(x - k_x p) - f(u - k_x p)| < \varepsilon$.

(ii) $u \geq k_x p + p$. Then $u < x + \delta < k_x p + p + p < k_x p + 2p$. We have $x - k_x p, u - k_x p \in [0, 2p]$ and $|(x - k_x p) - (u - k_x p)| = |x - u| < \delta$. Thus $|f(x) - f(u)| = |f(x - k_x p) - f(u - k_x p)| < \varepsilon$.

Combine these two cases and we deduce that $f$ is uniformly continuous on $\mathbb{R}$.