MATH 2050C Mathematical Analysis I
2019-20 Term 2

Hard problems in Chapter 4

4.1-12(c)

Suppose the limit exist, i.e. we assume

$$\lim_{x\to 0}(x + \operatorname{sgn}(x)) = a$$

for a real number a.

Now we choose $\epsilon = \frac{1}{4}$, as the limit exists, we can find $\delta > 0$ such that for all $0 < |x| < \delta$, we have

$$|x + \operatorname{sgn}(x) - a| < \epsilon$$

we choose a real number x_0 with $0 < x_0 < \delta$. Then let $x = x_0$ in above and then we get

$$x_0 + \operatorname{sgn}(x_0) - a < \epsilon \implies 1 - a < \frac{1}{4} \implies a > \frac{3}{4}$$

And we can also choose $x = -x_0$ and get

$$-x_0 - \operatorname{sgn}(-x_0) + a < \epsilon \implies a < -\frac{3}{4}$$

These two inequalities contridict with each other. Hence, the limit $\lim_{x\to 0}(x + \operatorname{sgn}(x))$ does not exist.

4.1-14

(a). If $L = 0$, by definition of limit, we have for each $\epsilon > 0$, we can find $\delta > 0$ such that if $0 < |x - c| < \delta$, then we have

$$\left|f(x)^2 - 0\right| < \epsilon^2$$

This will imply $|f(x)| < \epsilon$, or we can write it as $|f(x) - 0| < \epsilon$. Hence by the definition of limit,we get

$$\lim_{x\to c} f(x) = 0$$

(b). If $L \neq 0$, we can take $f(x) : \mathbb{R} \to \mathbb{R}$ by letting $f(x) = 1, x > c$, $f(x) = -1, x \leq c$. Then we get $\lim_{x\to c} f(x)^2 = 1$ but the limit $\lim_{x\to c} f(x)$ does not exist.

4.1-15

(a) For any $\epsilon > 0$, we can choose $\delta = \epsilon$, and then for any $0 < |x| < \delta$, we have

$$|f(x) - 0| = |x - 0| = |x| < \delta = \epsilon$$

if x is rational, and

$$|f(x) - 0) = |0 - 0| = 0 < \epsilon$$

if x is irrational. Hence

$$\lim_{x \to 0} f(x) = 0$$

(b) By Density Theorem 2.4.8 and Corollary 2.4.9, we can always find a sequence of rational numbers (x_n) such that $\lim x_n = c$ and a sequence of irrational numbers (y_n) such that $\lim y_n = c$. Then we will get $\lim f(x_n) = \lim x_n = c$ and $\lim f(y_n) = \lim 0 = 0$. But we note $c \neq 0$, this will lead to a contradiction if you assume $f(x)$ has a limit at c.

4.1-16

We can assume $I = (a, b)$ for some $a, b \in \mathbb{R}$. Then c will satisfies $a < c < b$. First, we notice c is an accumulate point of I.

Now let's suppose $\lim_{x \to c} f(x) = L$ and we will show that $\lim_{x \to c} f_1(x) = L$.

Indeed, by definition, for any $\epsilon > 0$, we can find $\delta > 0$ such that if $x \in \mathbb{R}$ satisfies $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon$. Since $V_\delta(c) \cap I \subset V_\delta(c)$, we know for any $x \in I \cap V_\delta(c)$, we also have $|f(x) - L| < \epsilon$. And since $f(x) = f_1(x)$ in I, we have $|f_1(x) - L| < \epsilon$ for $x \in V_\delta(c) \cap I$. Hence $\lim_{x \to c} f_1(x) = L$.

Conversely, we suppose $\lim_{x \to c} f_1(x) = L$. Then by definition we know that for any $\epsilon > 0$, we can find $\delta > 0$ such that if $x \in V_\delta(c) \cap I$, then $|f_1(x) - L| < \epsilon$. Now we can choose $\delta' = \min\{\delta, c - a, b - c\}$. Hence, for any $x \in V_{\delta'}(c)$, we know $x \in I$ by the choice of δ'. Then we have

$$|f(x) - L| = |f_1(x) - L| < \epsilon$$

since $f(x), f_1(x)$ agrees on I. Then by the definition of limit, we get $\lim_{x \to c} f(x) = L$.

4.1-17

The first part is essentially the same as above. If $\lim_{x \to c} f(x) = L$, then for any $\epsilon > 0$, we can find $\delta > 0$ with $|f(x) - L| < \epsilon$ for $x \in V_\delta(c)$. This will still holds for $x \in J \cap V_\delta(c)$, which will imply $|f_1(x) - L| < \epsilon$ for $x \in V_\delta(c) \cap J$. Hence $\lim_{x \to c} f_2(x) = L$.

But the converse might not true in general. For example, we take $f(x) = 1$ for $x \geq 0$ and $f(x) = 0$ for $x < 0$. Take $J = [0, 1]$. Clearly $f_2(x) = 1$ on J and hence the limit $\lim_{x \to 0} f_2(x) = 1$. But for $f(x)$. the limit $\lim_{x \to 0} f(x)$ does not exist.

4.2-10

Let choose $f(x) = (x), g(x) = -(x)$ and $c = 0$. Clearly, both limit $\lim_{x\to 0} f(x), \lim_{x\to 0} g(x)$ do not exist. But we note $f(x)+g(x) = 0$ all the time, hence $\lim_{x\to 0} f(x)+g(x)$ exists and equal to 0. And we note $f(x)g(x) = -1$ for all $x \neq 0$. Then we also know the limit $\lim_{x\to 0} fg$ exists and equal to -1.

4.2-11(d)

We just note that $-\sqrt{x} \leq \sqrt{x}\sin(\frac{1}{x^2}) \leq \sqrt{x}$ and $\lim_{x\to 0}\sqrt{x} = 0$, then by squeezing theorem, we have

$$\lim_{x\to 0}\sqrt{x}\sin(\frac{1}{x^2}) = 0$$

4.2-12

Let's suppose $L \neq 0$.

By the definition of limit, for a specific $\epsilon = \frac{|L|}{4}$, we can find $\delta > 0$ such that if $0 < |x| < \delta$, then $|f(x) - L| < \frac{|L|}{4}$.

Choose x_0 such that $0 < |x_0| < \delta$, then we set $x = y = \frac{x_0}{2}$ in the original equations and we get

$$\left|2f(\frac{x_0}{2}) - L\right| = |f(x_0) - L| < \frac{|L|}{4}$$

Then we get $\left|f(\frac{x_0}{2}) - \frac{L}{2}\right| < \frac{|L|}{8}$. In particular, we have

$$\left|f(\frac{x_0}{2}) - L\right| = \left|L - \frac{L}{2} + \frac{L}{2} - f(\frac{x_0}{2})\right| \geq \left|L - \frac{L}{2}\right| - \left|\frac{L}{2} - f(\frac{x_0}{2})\right| > \frac{3|L|}{8}$$

which contridicts with the face

$$\left|f(\frac{x_0}{2}) - L\right| < \frac{|L|}{4}$$

since $\frac{x_0}{2} \in V_\delta(0)$.

Next, let's show the limit $\lim_{x\to c} f(x)$ exists for every c. For any $\epsilon > 0$, we can find $\delta > 0$ with if $0 < |x| < \delta$, then $|f(x)| < \epsilon$ since $L = 0$. By replacing x by c and y by $x - c$, we have $f(x) = f(x-c) + f(c)$. Hence for any $0 < |x-c| < \delta$, we have

$$|f(x-c)| < \epsilon$$

by our choice of δ. Then we get

$$|f(x) - f(c)| = |f(x-c)| < \epsilon$$

Hence by definition of limit, we know that $\lim_{x\to c} f(x) = f(c)$.

Remark. This is a very famouse functional equations and there is a standard way to deal with it. From $f(x+y) = f(x) + f(y)$ and the fact $\lim_{x\to 0} f(x)$ exists, we can indeed show that $f(x) = cx$ for a real number c.