MATH 2050C Mathematical Analysis I
2019-20 Term 2

Hard problems in Chapter 2

2.1-23

Clearly the conclusion holds for $n = 1$. So by Mathematical induction, we only need to show when the conclusion holds for $n = k$, then it will hold for $n = k+1$.

First, if we have $a < b$, then by the assumption of induction, we will have $a^k < b^k$(conclusion holds for k). Then

$$a^{k+1} < ab^k < b \cdot b^k = b^{k+1}$$

On the other hand, if we have $a^{k+1} < b^{k+1}$, by the Order Properties, we have three cases, $a > b, a = b, a < b$. we need to rule out first two cases.

Indeed, if $a = b$, we will have $a^{k+1} = b^{k+1}$, which contradicts our assumtion. (You can also show this result by induction). And if $a > b$, then by our conclusion holds for k, we will have $a^k > b^k$ and similar as above, we can get $a^{k+1} > b^{k+1}$, which also contradicts our assumtion. Hence we can only have $a < b$. This will finish our proof.

2.1-25

$$c^{\frac{1}{m}} < c^{\frac{1}{n}} \underset{2.1-23}{\Longleftrightarrow} c^n < c^m \Longleftrightarrow 1 < c^{m-n}$$

So if $m > n$, then again by 2.1-23 and $1 < c$,

$$1^{m-n} < c^{m-n} \implies 1 < c^{m-n} \implies c^{\frac{1}{m}} < c^{\frac{1}{n}}$$

If we have $c^{\frac{1}{m}} < c^{\frac{1}{n}}$, clearly, we cannot have $m = n$ since $c^0 = 1$. We cannot have $m < n$ either since from above, we have

$$m < n \implies 1^{n-m} < c^{n-m} \implies 1 > c^{m-n} \implies c^{\frac{1}{m}} > c^{\frac{1}{n}}$$

a contradiction. So we only have $m > n$.

2.2-18

(a). Assume $a > b$ first, then

$$\frac{1}{2}(a+b+|a-b|) = \frac{1}{2}(a+b+a-b) = a = \max\{a,b\}$$

and

$$\frac{1}{2}(a+b-|a-b|) = \frac{1}{2}(a+b-a+b) = b = \min\{a,b\}$$

Similar for other direction.
(b). Let's assume a is the least one. That is, we assume $a \le b, a \le c$. Clearly, we will have $\min\{a,b,c\} = a$. And $\min\{\min\{a,b\}c\} = \min\{a,c\} = a$. So we have $\min\{a,b,c\} = \min\{\min\{a,b\}c\}$.

Similarly, if b is the least one, we have the same conclusion. For the case c is the least one, we just note that in either case $\min\{a,b\} = a$ or $\min\{a,b\} = b$, we will always have $\min\{a,c\} = c$ and $\min\{b,c\} = c$, which will implies

$$\min\{a,b,c\} = c = \min\{\min\{a,b\},c\}$$

2.3-10

Let $M_A, M_B > 0$ be the bounds for A, B respectively. Then for any $s \in A \cup B$, we will have $s \in A$ or $s \in B$, which implies $|s| \le M_A$ or $|s| \le M_B$, and hence $|s| \le \max\{M_A, M_B\}$, which shows $\max\{M_A, M_B\}$ is a bound for $A \cup B$.

Since the right side has only two elements, we can easlity proof that $\sup\{\sup A, \sup B\} = \max\{\sup A, \sup B\}$.

Let's proof $\max\{\sup A, \sup B\}$ is a supremum of $A \cup B$. Clearly, for any $s \in A \cup B$, we will have $s \le \sup A$ or $s \le \sup B$, which implies $s \le \max\{\sup A, \sup B\}$. So we know that $\max\{\sup A, \sup B\}$ is an upper bound of $A \cup B$.

On the other hand, for any upper bound u of $A \cup B$, we know that u is an upper bound of A, and hence $u \ge \sup A$. Again, we know that u is an upper bound of B since $B \subset A \cup B$ which implies $u \ge \sup B$. Then we will have $u \ge \max\{\sup A, \sup B\}$. So $\max\{\sup A, \sup B\}$ is the least upper bound of $A \cup B$, which implies $\sup(A \cup B) = \max\{\sup A, \sup B\}$

2.4-8

We will only proof the first statement, since the second one is similar with the first one.

Let's write $a = \sup\{f(x) : x \in X\}, b = \sup\{g(x) : x \in X\}$. Now for any element in $s \in \{f(x) + g(x) : x \in X\}$, we know s can be written as

$$s = f(x_0) + g(x_0)$$

for some $x_0 \in X$. By the definition of supremum, we have

$$f(x_0) \le a, \quad g(x_0) \le b$$

Hence, we get

$$s = f(x_0) + g(x_0) \leq a + b$$

for each $s \in \{f(x) + g(x) : x \in X\}$. This shows $a + b$ is an upper bound, which we can write as

$$\sup\{f(x) + g(x) : x \in X\} \leq \sup\{f(x) : x \in X\} + \sup\{g(x) : x \in X\}$$

Example of strict inequalities. Take $X = \mathbb{R}$, $f(x) = \sin x, g(x) = \cos x$. Clearly, we have $\sup\{\sin x + \cos x : x \in X\} = \sqrt{2}$ and $\sup\{\sin x : x \in X\} = \sup\{\cos x : x \in X\} = 1$, which shows

$$\sup\{f(x) + g(x) : x \in X\} = \sqrt{2} < 2 = \sup\{f(x) : x \in X\} + \sup\{g(x) : x \in X\}$$

2.4-12

Based on the symmetric of x, y, we can only proof the first part of equalities, i.e. we will show

$$\sup_{x,y} h(x,y) = \sup_x \sup_y h(x,y)$$

Or we can write it in another form

$$\sup\{h(x,y) : x \in X, y \in Y\} = \sup\{F(x) : x \in X\}$$

First, we note for any $h(x,y) \in \{h(x,y) : x \in X, y \in Y\}$, we will have

$$h(x,y) \leq \sup\{h(x,y) : y \in Y\} = F(x)$$

by the definition of supremum. Again by definition, we have $F(x) \leq \sup\{F(x) : x \in X\}$.

This shows $\sup\{F(x) : x \in X\}$ is an upper bound of $\sup\{h(x,y) : x \in X, y \in Y\}$. So the thing left is we need to show it is also the least one. Indeed, we fix any $s < \sup\{F(x) : x \in X\}$. Again from the definition of supremum, we can find an element in $\{F(x) : x \in X\}$, says $F(x_0)$, such that $F(x_0) > s$. But we notice that

$$\{h(x_0, y) : y \in Y\} \subset \{h(x,y) : x \in X, y \in Y\}$$

so at least we will have

$$F(x_0) = \sup\{h(x_0, y) \in Y\} \leq \sup\{h(x,y) : x \in X, y \in Y\}$$

This will imply $s < \sup\{h(x,y) : x \in X, y \in Y\}$. Hence s is not an upper bound of $\{h(x,y) : x \in X, y \in Y\}$. Thus, we get

$$\sup\{h(x,y) : x \in X, y \in Y\} = \sup\{F(x) : x \in X\}$$

2.5-10

We will just follow the proof in textbook.

By definition, we have $\eta = \inf\{b_n : n \in \mathbb{N}\}$ if we set $I_n = [a_n, b_n]$. Clearly we will have $\eta < b_n$ for all n by the definition of infimum. We will show that $\eta \geq a_n$ for all n next.

Indeed, for any a_k, we claim a_k is a lower bound of $\{b_n : n \in \mathbb{N}\}$. This is because, for any b_n, if $n \leq k$, we will have $a_k \leq b_k \leq b_n$ by the definition of nest intervals. And if $n > k$, we have $a_k \leq a_n \leq b_n$ by the definition also. Hence a_k is a lower bound of $\{b_n : n \in \mathbb{N}\}$, and it implies $\eta \geq a_k$. Hence for any n, we have $a_n \leq \eta \leq b_n$, i.e. $\eta \in I_n$. So $\eta \in \bigcap_{n=1}^{\infty} I_n$.

At least, we show $[\xi, \eta] = \cap_{n=1}^{\infty} I_n$. Perhaps you need to show $\xi \leq \eta$ first to make sure it is well defined. But this is clear by $\sup\{a_n : n \in \mathbb{N}\} \leq \inf\{b_n : n \in \mathbb{N}\}$ since for every $m, n \in \mathbb{N}$, we have $a_m \leq b_n$. (By the definition of nested intervals). So for any $x \in [\xi, \eta]$ and definition of supremum and infimum, we have

$$a_n \leq \xi \leq x \leq \eta \leq b_n$$

which implies $x \in \bigcap_{n=1}^{\infty} I_n$. On the other hand, for any $y \in \bigcap_{n=1}^{\infty} I_n$, we have $a_n \leq y \leq b_n$ for every n, and by the properties of supremum and infimum, we have

$$\xi \leq y \leq \eta$$

Hence, we find

$$[\xi, \eta] = \bigcap_{n=1}^{\infty} I_n$$