MATH 2050C Mathematical Analysis I
2019-20 Term 2

Solution to Problem Set 9

4.2-4

Denote $f(x) := \cos\frac{1}{x}, x \in \mathbb{R} \setminus \{0\}$, $a_n = \frac{1}{2n\pi}, n \in \mathbb{N}$ and $b_n = \frac{1}{(2n+\frac{1}{2})\pi}, n \in \mathbb{N}$. Note that $a_n, b_n \neq 0, \forall n \in \mathbb{N}$ and that (a_n) and (b_n) are convergent sequences with common limit 0. Suppose that $\lim_{x\to 0} f(x) = L$ exist, which implies that $L = \lim(f(a_n)) = \lim(f(b_n))$ by Theorem 4.1.8(b) and Theorem 4.1.5. But $f(a_n) = \cos 2n\pi = 1$ and $f(b_n) = \cos(2n + \frac{1}{2})\pi = 0$ for any $n \in \mathbb{N}$. Thus $\lim f(a_n) = 1$ while $\lim f(b_n) = 0$, a contradiction.
Denote $g(x) := x\cos\frac{1}{x}, x \in \mathbb{R} \setminus \{0\}$. Note that $|\cos y| \leq 1, \forall y \in \mathbb{R}$. Given $\varepsilon > 0$, set $\delta = \varepsilon$. For any x satisfying $0 < |x| < \delta$,

$$|g(x)| = |x|\,|\cos(1/x)| \leq |x| < \varepsilon.$$

Since ε is arbitrary, $\lim_{x\to 0} x\cos\frac{1}{x} = 0$.

4.2-5

By the supposition, there exists $\delta_1 > 0$ and $M > 0$ so that $|f(x)| < M, \forall x \in (c - \delta_1, c + \delta_1)$. Given $\varepsilon > 0$, there exists $\delta_2 > 0$ so that $|g(x)| < \varepsilon/M, \forall x \in (c - \delta_2, c + \delta_2) \setminus \{c\}$. Set $\delta = \min\{\delta_1, \delta_2\}$. For x satisfying $0 < |x - c| < \delta$, we have

$$|f(x)g(x)| < M \cdot (\varepsilon/M) = \varepsilon.$$

4.2-9

(a). Let $h(x) = f(x) + g(x)$, then since both limit $\lim_{x\to c} f, \lim_{x\to c} h$ exists, so does the limit $\lim_{x\to c} h - f$. This limit is exactly $\lim_{x\to c} g$.
(b). Take $f(x) = x - c$ and $g(x) = \frac{1}{x-c}$. Clearly, we know both limits $\lim_{x\to c} x - c = 0$ and $\lim_{x\to c} fg = \lim_{x\to c} 1$ exists. But the limit $\lim_{x\to c} g(x) = \lim_{x\to c} \frac{1}{x-c}$ does not exist.

4.2-15

We consider the follwing two cases.

First, if $\lim_{x\to c} f = 0$, we just need to proof $\lim_{x\to c} \sqrt{f} = 0$. Indeed, by the definition of limit $\lim_{x\to c} f = 0$, for any $\epsilon > 0$, we can choose $\delta > 0$ such that if $0 < |x - c| < \delta$, then

$$|f(x)| < \epsilon^2$$

Then we have $|\sqrt{f(x)}| < \epsilon$. Hence $\lim_{x\to c} \sqrt{f} = 0$.

Second, if $\lim_{x\to c} f > 0$, we write $A = \lim_{x\to c} f$. Again, for any $\epsilon > 0$, we can choose $\delta > 0$, and if $0 < |x - c| < \delta$, then we have

$$|f(x) - A| < \epsilon\sqrt{A}$$

Then we get for any $0 < |x - c| < \delta$,

$$|\sqrt{f}(x) - \sqrt{A}| = \frac{|f(x) - \sqrt{A}|}{\sqrt{f(x)} + \sqrt{A}} < \frac{\epsilon\sqrt{A}}{\sqrt{A}} = \epsilon$$

Hence, $\lim_{x\to c} \sqrt{f} = \sqrt{A} = \sqrt{\lim_{x\to c} f}$.