MATH 2050C Mathematical Analysis I
2019-20 Term 2

Solution to Problem Set 8

3.5-2(a)

For any $\epsilon > 0$, we can choose $H > \frac{2}{\epsilon}$. Then for any $n, m \geq H$, we have

$$\left|\frac{n+1}{n} - \frac{m+1}{m}\right| = \left|\frac{1}{n} - \frac{1}{m}\right| \leq \frac{1}{n} + \frac{1}{m} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Hence, $\left(\frac{n+1}{n}\right)$ is a Cauchy sequence.

3.5-3(c)

Choose a special $\epsilon = 1$, we will show that for any integer, $H > 0$, there is $m, n \geq H$ such that $|\ln m - \ln m| \geq \epsilon = 1$. Indeed, for any integer $H > 0$, we choose $n = H$, and $m = 3H$, then we have

$$|\ln m - \ln n| = |\ln(3H) - \ln H| = \ln 3 > 1 = \epsilon$$

Hence, $(\ln n)$ is not a Cauchy sequence.

3.5-7

We also choose a special $\epsilon > 0$, namely $\epsilon = \frac{1}{2}$. Then by definition, we can find an integer $H > 0$, such that for any $n, m \geq H$, we have $|x_n - x_m| < \epsilon = \frac{1}{2}$. Since x_n, x_m are all integers, we must have $x_n = x_m$, which means $x_m =$ a fixed integer for all $n \geq H$. Hence, (x_n) is ultimately constant.

3.5-10

To show the convergence, it suffices to verify that (x_n) is a contractive sequence and apply Theorem 3.5.8. From the iteration formula $x_n = \frac{1}{2}(x_{n-2} + x_{n-1})$,

$$\begin{aligned} |x_{n+2} - x_{n+1}| &= \left| \frac{1}{2}(x_{n+1} + x_n) - x_{n+1} \right| \\ &= \left| \frac{1}{2}(x_{n+1} - x_n) \right| \\ &\leq \frac{1}{2} \left| x_{n+1} - x_n \right|, \end{aligned}$$

which verifies the condition of contraction.
To evaluate the limit, by the iteration formula again and for $n \geq 2$,

$$x_n - x_{n-1} = -\frac{1}{2}(x_{n-1} - x_{n-2}) = (-\frac{1}{2})^2(x_{n-2} - x_{n-3}) = \cdots = (-\frac{1}{2})^{n-2}(x_2 - x_1).$$

Combining with the identity

$$x_n = x_1 + \sum_{i=2}^{n}(x_i - x_{i-1}),$$

we obtain

$$\begin{aligned} x_n &= x_1 + \sum_{i=2}^{n}(x_i - x_{i-1}) \\ &= x_1 + \sum_{i=2}^{n}(-\frac{1}{2})^{i-2}(x_2 - x_1) \\ &= x_1 + (\frac{1 - (-\frac{1}{2})^{n-1}}{1 - (-\frac{1}{2})})(x_2 - x_1) \\ &= x_1 + \frac{2}{3}(1 - (-\frac{1}{2})^{n-1})(x_2 - x_1). \end{aligned}$$

Thus

$$\lim x_n = \lim[x_1 + \frac{2}{3}(1 - (-\frac{1}{2})^{n-1})(x_2 - x_1)] = x_1 + \frac{2}{3}(x_2 - x_1) = \frac{x_1 + 2x_2}{3}.$$

4.1-8

For any $\epsilon > 0$, we can choose $\delta < \min\{\epsilon\sqrt{c}, c\}$, then for any $0 < |x - c| < \delta$, we have $x > 0$ first, then

$$\left|\sqrt{x} - \sqrt{c}\right| = \left| \frac{x - c}{\sqrt{x} + \sqrt{c}} \right| < \frac{\delta}{\sqrt{c}} < \epsilon$$

Then

$$\lim_{x \to c} \sqrt{x} = \sqrt{c}$$

4.1-9(d)

For any $\epsilon > 0$, we choose $\delta = \min\{1, \frac{\epsilon}{3}\}$, and if $|x-1| < \delta$, we have

$$\left|\frac{x^2-x+1}{x+1} - \frac{1}{2}\right| = \left|\frac{(x-1)(2x-1)}{2(x+1)}\right| < \frac{\delta|2x-1|}{2|x+1|} \leq \frac{5\delta}{2} \leq \epsilon$$

Hence,

$$\lim_{x\to 1} \frac{x^2-x+1}{x+1} = \frac{1}{2}$$

4.1-12(d)

Denote $g(x) := \sin\frac{1}{x^2}, x \in \mathbb{R}\backslash\{0\}$, $a_n = \frac{1}{\sqrt{2n\pi}}, n \in \mathbb{N}$ and $b_n = \frac{1}{\sqrt{(2n+\frac{1}{2})\pi}}, n \in \mathbb{N}$. Note that $a_n, b_n \neq 0, \forall n \in \mathbb{N}$ and that (a_n) and (b_n) are convergent sequences with common limit 0. Suppose that $\lim_{x\to 0} g(x) = L$ exist, which implies that $L = \lim(g(a_n)) = \lim(g(b_n))$ by Theorem 4.1.8(b) and Theorem 4.1.5. But $g(a_n) = \sin 2n\pi = 0$ and $g(b_n) = \sin(2n+\frac{1}{2})\pi = 1$ for any $n \in \mathbb{N}$. Thus $\lim g(a_n) = 0$ while $\lim g(b_n) = 1$, a contradiction.

4.1-13

For any $\epsilon > 0$, since

$$\lim_{x\to 0} f(x) = L$$

we can find $\delta > 0$, such that if x satisfying $0 < |x| < \delta$, then we have $|f(x)-L| < \epsilon$. Now for any x satisfying $0 < |x| < \frac{\delta}{a}$, we know that $0 < |ax| < \delta$. Hence, by the definition of $g(x)$, we have

$$|g(x) - L| = |f(ax) - L| < \epsilon$$

Hence, we get the existence of $\delta' = \frac{\delta}{a}$ such that if $0 < |x| < \delta'$, then we have

$$|g(x) - L| < \epsilon$$

Then we get

$$\lim_{x\to 0} = L$$