MATH 2050C Mathematical Analysis I
2019-20 Term 2

Solution to Problem Set 7

3.4-2

As in textbook, we show that $(c^{\frac{1}{n}})$ is an increasing sequence for $0 < c < 1$. Indeed, $c^{\frac{1}{n}} < c^{\frac{1}{n+1}} \leftrightarrow c^{n+1} < c^n \leftrightarrow c < 1$.

Clearly this sequence is bounded above by 1. So using Monotone Convergence Theorem, we can assume $x = \lim c^{\frac{1}{n}}$. We note that $(c^{\frac{1}{2n}})$ is a subsequence of $(c^{\frac{1}{n}})$ and hence $x = \lim c^{\frac{1}{2n}}$. Beside, by the operation of limit, we have $x = \lim c^{\frac{1}{2n}} = (\lim c^{\frac{1}{n}})^{\frac{1}{2}} = \sqrt{x}$. Hence we get $x^2 = x \implies x = 1$ or $x = 0$. But $x = 0$ is impossible since $c^{\frac{1}{n}} > c > 0$ for all n and hence the limit $x \geq c$ at least. so we get

$$\lim c^{\frac{1}{n}} = 1$$

3.4-4(b)

Take $n = 4k$, then we have $\sin \frac{4k\pi}{4} = \sin k\pi = 0$. But if we choose $n = 8k+2$, then $\sin \frac{(8k+2)\pi}{4} = \sin \frac{\pi}{2} = 1$. Hence $(\sin \frac{n\pi}{4})$ has two subsequences which converges to different values, 0 and 1. Hence it is divergence by **Divergence Criterial**.

3.4-6

(a). First, we have

$$(n+1)^{\frac{1}{n+1}} < n^{\frac{1}{n}} \Leftrightarrow (n+1)^n < n^{n+1} \Leftrightarrow \left(\frac{n+1}{n}\right)^n < n$$

So by Example 3.3.6, we know that $\left(1 + \frac{1}{n}\right)^n$ is bounded above by 3 strictly and hence $\left(1 + \frac{1}{n}\right)^n < n$ is valid for $n \geq 3$. So for $n \geq 3$, we know that the sequence (x_n) is indeed decreasing and hence the limit exists. So we can assume $x = \lim x_n$.

(b). Indeed, we have

$$x = \lim x_{2n} = \lim(2^{\frac{1}{2n}} n^{\frac{1}{2n}}) = \lim 2^{\frac{1}{2n}} \times \left(\lim n^{\frac{1}{n}}\right)^{\frac{1}{2}} = 1 \times x^{\frac{1}{2}}$$

So we get $x = x^2 \implies x = 0$ or $x = 1$. But $x = 0$ cannot happen since $x_n \geq 1$ all the time. So we get $\lim n^{\frac{1}{n}} = 1$.

3.4-9

Since some subsequence of (x_n) converges to 0, the only possible limit of (x_n) is 0. On the contrary, suppose that (x_n) is divergent. Then (x_n) dose not converges to 0. By Theorem 3.4.4(iii), there exists $\epsilon_0 > 0$ and a subsequence (x_{n_k}) of (x_n) such that $|x_{n_k} - 0| > \epsilon_0$ for all $k \in \mathbb{N}$. By assumption, we can find a further subsequence (which by abuse of notation, we still denote by (x_{n_k}) of (x_{n_k})) converging to 0. So, we have found a subsequence (x_{n_k}) of (x_n) which is converging to 0 and $|x_{n_k}| \geq \varepsilon_0$ for all $k \in \mathbb{N}$ at the same time. This is a contradiction.

3.4-11

Let $x = \lim(-1)^n x_n$. Then any subsequence of it will also converge to x, that is, we have

$$x = \lim(-1)^{2n} x_{2n} = \lim x_{2n}$$

and

$$x = \lim(-1)^{2n+1} x_{2n+1} = -\lim x_{2n+1}$$

Since $x_n \geq 0$, we have $\lim x_{2n} \geq 0, \lim x_{2n+1} \geq 0$. Hence $x \geq 0, x \leq 0 \implies x = 0 \implies \lim(-1)^n x_n = 0$. By exercise 3.1-8, and $x_n \geq 0$,

$$\lim(-1)^n x_n = 0 \Longleftrightarrow \lim |x_n| = 0 \Longleftrightarrow \lim x_n = 0$$

That means (x_n) converges.